Math 445 — Problem Set #2
Due: Friday, September 8 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Let a, b, c be integers. Show that if a and b are coprime, a divides c, and b divides c, then ab divides c.

(2) Find all solutions to the equation $x^2 + [4]x = [5]$ in $\mathbb{Z}_8$ by trial and error (plugging in all possible values). Use this to find all integer solutions to $x^2 + 4x \equiv 5 \pmod 8$.

(3) Given integers $a_1, \ldots, a_m$, the **greatest common divisor** of $a_1, \ldots, a_m$ is the largest integer that divides all of them.
  (a) Compute $\gcd(12, 18, 42)$.
  (b) Prove or disprove: If $\gcd(a, b, c) = 1$, then some pair of the numbers a, b, c is coprime.

(4) *Use the methods we have developed in class* to solve the following:
  (a) Find all integer pairs (x, y) such that $275x - 126y = 9$.
  (b) Find the inverse of $[126]$ in $\mathbb{Z}_{275}$.
  (c) Find the smallest positive integer x such that

$$x \equiv 7 \pmod{126} \quad \text{and} \quad x \equiv 8 \pmod{275}.$$

(5) Solving linear equations in $\mathbb{Z}_n$: Let a, b, n be integers, with $n > 0$.
  (a) Show that $[a]x = [b]$ has a solution x in $\mathbb{Z}_n$ if and only if $\gcd(a, n)$ divides b.
  (b) Show that if $[a]x = [b]$ has a solution x in $\mathbb{Z}_n$, then there are exactly $\gcd(a, n)$ distinct solutions.
  (c) Solve the equation $[20][x] + [17] = [29]$ in $\mathbb{Z}_{36}$.

The remaining problems are only required for Math 845 students, though all are encouraged to think about them.

(6) Solve the equation $8x + 25y + 15z = 19$ over $\mathbb{Z}$.