1 Definition/theorem/axiom statements

1. State the definition of an *upper bound* for a set of real numbers S.

A number b is an upper bound for S if for all $x \in S$, $\cancel{x} \leq b$.

2. State the *Completeness Axiom*.

Every nonempty bounded above subset of $\mathbb{R}$ has a supremum.

3. State the definition for a sequence $\{a_n\}_{n=1}^{\infty}$ to *diverge to* $+\infty$.

For every $M \in \mathbb{R}$, there exists $N \in \mathbb{R}$ such that for all natural numbers $n > N$, $a_n > M$.

2 True or false

Determine whether each of the statements below is *true* or *false*, and justify your choice with a short argument or example.

1. Let x be a real number. If $x^2 - \frac{1}{5}$ is irrational, then x is irrational.

TRUE

If x is rational, then $x^2 = x \cdot x$ is rational, and since $-\frac{1}{5}$ is rational, $x^2 - \frac{1}{5}$ is ~~too~~.

2. Every increasing bounded below sequence converges.

FALSE

Take $\{n\}_{n=1}^{\infty}$.
It is bounded below by 0 but diverges, since it's not bounded above.

3. If S is a set of real numbers such that

- $-7 \in S$, and
- every element of S is negative,

then S has a supremum.

TRUE

S is nonempty since $-7 \in S$ and is bounded above by 0, so S has a supremum by completeness.

4. There is a sequence of positive numbers that converges to $-\frac{1}{100}$.

FALSE

If $\{a_n\}_{n=1}^{\infty}$ were such a sequence, take $\varepsilon = \frac{1}{100}$, then $\exists N: \forall n > N$,

$$\left| a_n - \frac{-1}{100} \right| < \frac{1}{100}$$

$$\Downarrow$$

$$\frac{-2}{100} < a_n < \frac{-1}{100} + \frac{1}{100} = 0,$$

so $a_n < 0$, a contradiction.

3 Proofs

1. Use the definition (and not any theorems about limits of sequences) to prove that the sequence $\left\{\frac{2n-7}{5n}\right\}_{n=1}^{\infty}$ converges to $\frac{2}{5}$.

Let $\varepsilon > 0$.

Take $N = \frac{7}{5\varepsilon}$. Then for $n > N$, we have

$$\left|\frac{2n-7}{5n} - \frac{2}{5}\right| = \left|\frac{-7}{5n}\right| = \frac{7}{5n} < \frac{7}{5N} = \varepsilon.$$

This shows ~~that~~ $\left\{\frac{2n-7}{5n}\right\}_{n=1}^{\infty}$ conv. to $\frac{2}{5}$.

2. Prove that 3 is the supremum of the set

$$\{z \in \mathbb{R} \mid z \text{ is irrational and } z < 3\}.$$

First, we have that 3 is an upper bound, since any element of the set is less than 3 by definition.

To see there is no smaller upper bound, let $b < 3$. Then by density of irrat'ls, there is an irrat'l z s.t. $b < z < 3$. This z is an element of the set, so b is not an upper bound. It follows that if b is an upper bound for the set then $b \geq 3$.

3. (a) Prove that for every natural number n, $n < 2^n$.

By induction on $n \in \mathbb{N}$.

Let $n=1$; then we have $1 < 2^1$, which is true.

Assume that for some $k \in \mathbb{N}$, $k < 2^k$. Then

$$k+1 < 2^k + 1 \leq 2^k + 2^k = 2^{k+1}.$$

By induction it holds for all $n \in \mathbb{N}$.

(b) Prove[1] that $\left\{7 - \frac{1}{2^n}\right\}_{n=1}^{\infty}$ converges to 7.

Since $n < 2^n$, $\frac{1}{n} > \frac{1}{2^n}$, so

$$7 - \frac{1}{n} < 7 - \frac{1}{2^n} < 7 \text{ for all } n.$$

Since $\{7 - \frac{1}{n}\}_{n=1}^{\infty}$ and $\{7\}_{n=1}^{\infty}$ both converge to 7,

$\{7 - \frac{1}{2^n}\}_{n=2}^{\infty}$ converges to 7.

[1] You may use any theorems about sequences we proved in class, including the Squeeze Theorem. You may also use the conclusion of part (a) even if you did not prove it.

Bonus problem

Prove or disprove: If

- $\{a_n\}_{n=1}^{\infty}$ converges to L,
- $L < 0$, and
- $\{b_n\}_{n=1}^{\infty}$ diverges to $+\infty$,

then $\{a_n b_n\}_{n=1}^{\infty}$ diverges to $-\infty$.

Let $M \in \mathbb{R}$.

By definition of converges to L applied to $-L/2 > 0$, there is some $N_1 \in \mathbb{R}$ s.t. $\forall n > N_1$, $|a_n - L| < \frac{-L}{2}$, so

$$a_n < L - \frac{L}{2} = \frac{L}{2} < 0.$$

By definition of diverges to $+\infty$ applied to $\frac{2M}{L}$, there is some $N_2 \in \mathbb{R}$ s.t.

$\forall n > N_2$, $b_n > \max\{\frac{2M}{L}, 0\}$

Then set $N = \max\{N_1, N_2\}$.

If $n > N$, then $n > N_1$ & $n > N_2$, so

$$a_n b_n < \frac{L}{2} \cdot \frac{2M}{L} = M.$$

(since a_n is negative and b_n is positive)