Name: Solutions

Instructions:

- Take your time and read the instructions and questions carefully.
- You may use any results that we have proven in lecture, in the groupwork in class, or on the homework, except
  - if it is exactly the thing you are asked to show, or
  - if the problem specifies otherwise.

  If you are using a result, you should clearly state what result you are using.
- Good luck!

Problem	Points	Score
1.1	7	
1.2	7	
1.3	7	
2.1	7	
2.2	7	
2.3	7	
2.4	7	
2.5	11	
3.1	20	
3.2	20	
Bonus		
Total	100	

1 Definition/theorem statements

1. Let $f(x)$ be a function and a, L be real numbers. State the definition for *the limit of $f(x)$ as x approaches a* to equal L.

For any $\varepsilon > 0$ there is some $\delta > 0$ such that if $0 < |x - a| < \delta$ then f is defined at x and $|f(x) - L| < \varepsilon$.

2. State the *Bolzano-Weierstrass Theorem.* (Either the Theorem or Corollary that went by these names is OK.)

Every sequence has a monotone subsequence.

— or —

Every bounded sequence has a convergent subsequence.

3. State the *Completeness Axiom.*

Every nonempty bounded-above set of real numbers has a supremum.

2 True or false

Determine whether each of the statements below is *true* or *false*, and justify your choice with a short argument or example.

1. Every sequence has a convergent subsequence.

False:
every subsequence of $\{n\}_{n=1}^{\infty}$ diverges (to $+\infty$).

2. If $2 < f(x) < 7$ for all x, then $\lim_{x \to 0} xf(x) = 0$.

True:
$2 < f(x) < 7 \Rightarrow |f(x)| < 7$
$\Rightarrow |x f(x)| < 7|x|$
so
$-7|x| \leq x f(x) \leq 7|x|$.
Since $\lim_{x \to 0} -7|x| = \lim_{x \to 0} 7|x| = 0$,
$\lim_{x \to 0} x f(x) = 0$ by Squeeze Thm.

3. If $\{a_n\}_{n=1}^{\infty}$ is a convergent sequence and $\{a_{2k}\}_{k=1}^{\infty}$ converges to 3, then $\{a_{2k+1}\}_{k=1}^{\infty}$ converges to 3.

True: Say $\{a_n\}_{n=1}^{\infty}$ converges to L. Then $\{a_{2k}\}_{k=1}^{\infty}$ converges to L, so $L=3$, and $\{a_{2k+1}\}_{k=1}^{\infty}$ conv. to L, so it conv. to 3.

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4. If $f(x)$ and $g(x)$ are functions, and $\lim_{x \to a} g(x) = 0$, then $\lim_{x \to a} \frac{f(x)}{g(x)}$ does not exist.

False: Take $f(x)=g(x)=x$ (and $a=0$); then $\frac{f(x)}{g(x)} = 1$ (on domain $\mathbb{R}\setminus\{0\}$), so $\lim_{x \to 0} g(x) = 0$ but $\lim_{x\to 0} \frac{f(x)}{g(x)}$ exists (and $=1$).

5. Both parts of this problem are about the sequence $\{a_n\}_{n=1}^{\infty}$ where

$$a_n = n\sqrt{2} - \lfloor n\sqrt{2} \rfloor,$$

and $\lfloor x \rfloor$ denotes[1] the largest *integer* less than or equal to x.

(a) $\{a_n\}_{n=1}^{\infty}$ has a *convergent* subsequence.

True: $\lfloor n\sqrt{2} \rfloor \leq n\sqrt{2} < \lfloor n\sqrt{2} \rfloor + 1$

$\Rightarrow \quad 0 \leq n\sqrt{2} - \lfloor n\sqrt{2} \rfloor < 1$

so $\{a_n\}_{n=1}^{\infty}$ is bounded between 0 & 1. By BW, it has a conv. subseq.

(b) $\{a_n\}_{n=1}^{\infty}$ has a *constant* subsequence.

False: If $m<n$ and $a_m = a_n$, then

$$m\sqrt{2} - \lfloor m\sqrt{2} \rfloor = n\sqrt{2} - \lfloor n\sqrt{2} \rfloor$$

$$\Rightarrow (n-m)\sqrt{2} = \lfloor n\sqrt{2} \rfloor - \lfloor m\sqrt{2} \rfloor \in \mathbb{Z},$$

so $$\sqrt{2} = \frac{\lfloor n\sqrt{2} \rfloor - \lfloor m\sqrt{2} \rfloor}{n-m} \in \mathbb{Q},$$

a contradiction.

[1] In particular, $\lfloor x \rfloor$ is an integer and $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$ for any real number x.

3 Proofs

1. *Use the definition* of limit to show that the function f with domain $\mathbb{R} \setminus \{0\}$ given by the rule
$$f(x) = \begin{cases} 2x+1 & \text{if } x > 0 \\ -x+1 & \text{if } x < 0 \end{cases},$$
has the following limit:
$$\lim_{x \to 0} f(x) = 1.$$

Let $\varepsilon > 0$.

Take $\delta = \varepsilon/2$.

Given x such that $0 < |x| < \delta$,

if $x > 0$, then $|f(x) - 1| = |2x + 1 - 1|$

$= |2x| < 2\delta = \varepsilon$ (and $f(x)$ is defined);

if $x < 0$, then $|f(x) - 1| = |-x + 1 - 1|$

$= |-x| < \delta = \varepsilon/2 < \varepsilon$ ($f(x)$ defined);

~~so~~ for any such x, $f(x)$ is defined

and $|f(x) - 1| < \varepsilon$.

This shows that the limit is 1.

2. Prove that if $f(x)$ is continuous at a, and $f(a)$ is positive, then there is some open interval $(a-\delta, a+\delta)$ containing a such that $f(x)$ is positive for all $x \in (a-\delta, a+\delta)$.

Take $\varepsilon = f(a)$, which is positive by assumption. By definition of continuous at a, there is some $\delta > 0$ such that ~~that~~ if $|x-a|<\delta$ then $|f(x)-f(a)| < f(a)$.

For this δ, if $x \in (a-\delta, a+\delta)$, then $|x-a|<\delta$, so $|f(x)-f(a)| < f(a)$, hence $-f(a) < f(x) - f(a)$ and $f(x) > 0$, as required.

Bonus problem

Prove or disprove:[2] Let f be a function defined on $\mathbb{R}$. Then f is continuous at a if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$.

($\Rightarrow$) Let $\{x_n\}_{n=1}^{\infty}$ converge to a.
Let $\varepsilon > 0$. By def. of cts. at a, there is some $\delta > 0$ such that if $|x-a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.
By def. of $\{x_n\}_{n=1}^{\infty}$ conv. to a applied to $\delta > 0$, there is some $N \in \mathbb{R}$ such that if $n > N$ then $|x_n - a| < \delta$. For this N, if $n > N$, then we have $|f(x_n) - f(a)| < \varepsilon$.
This shows that $\{f(x_n)\}_{n=1}^{\infty}$ conv. to $f(a)$.

($\Leftarrow$) Assume that for every seq $\{x_n\}_{n=1}^{\infty}$ that converges to a, then $\{f(x_n)\}_{n=1}^{\infty}$ ~~converges~~ converges to $f(a)$. Then if $\{x_n\}_{n=1}^{\infty}$ is a seq that converges to a and satisfies $x_n \neq a$ for all n, $\{f(x_n)\}_{n=1}^{\infty}$ converges to a still. By the Corollary on limits & sequences, $\lim_{x \to a} f(x) = f(a)$.
By thm on limits & cty., f is cts at a.

[2] As noted in the instructions, you can use any result we have proved in class, but if you want to use a result, be sure you are stating it totally precisely and accurately.

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