Math 325-002 — Problem Set #9
Due: Thursday, November 10 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Using any theorems and previously computed examples about limits, compute

$$\lim_{x\to 0}\left(\frac{|x|}{x^2+5}+x\cos\left(\frac{\sin(x)}{x^9}\right)\right).$$

(2) Using just the $\epsilon-\delta$ definition of limit, show that[1] for any $a>0$, $\lim_{x\to a}\sqrt{x}=\sqrt{a}$.

(3) Let $f(x)$ be the function with domain $\mathbb{R}$ given by the rule

$$f(x)=\begin{cases}1 & \text{if } x\in\mathbb{Z}\text{, and}\\ -1 & \text{if } x\notin\mathbb{Z}.\end{cases}$$

Prove that for any $a\in\mathbb{R}$, we have $\lim_{x\to a}f(x)=-1$.

(4) Let $f(x)$ be the function with domain $\mathbb{R}$ given by the rule

$$f(x)=\begin{cases}x & \text{if } x\in\mathbb{Q}\text{, and}\\ 0 & \text{if } x\notin\mathbb{Q}.\end{cases}$$

(a) Prove that $\lim_{x\to 0}f(x)=0$.
(b) Prove that if $a\neq 0$, then $\lim_{x\to a}f(x)$ does not exist.

Definition: Let f be a function and $a\in\mathbb{R}$. We say that *the limit of $f(x)$ as x approaches a from the right is L* provided:

For every $\epsilon>0$, there is some $\delta>0$ such that for all x satisfying $a<x<a+\delta$, we have that f is defined at x and also that $|f(x)-L|<\epsilon$.

In this case, we write $\lim_{x\to a^+}f(x)=L$.

(5) Use the definition to prove that $\lim_{x\to 0^+}\sqrt{x}=0$.

[1]Hint: You may want to use that $|\sqrt{x}-\sqrt{a}|=\dfrac{|x-a|}{|\sqrt{x}+\sqrt{a}|}\le\dfrac{|x-a|}{|\sqrt{a}|}$.