Math 325-002 — Problem Set #7
Due: Thursday, October 27 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) For each of the following, give an explicit example as indicated; no proofs are necessary.
   (a) A sequence that has a subsequence that converges to 1, another subsequence that converges to 2, and a third subsequence that converges to 3.
   (b) A sequence that has one sequence that is monotone and converges to 0 and another subsequence that is monotone and diverges to $+\infty$.

(2) Prove that if $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$, then every subsequence of $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$.

(3) Define a sequence $\{a_n\}_{n=1}^{\infty}$ recursively by $a_1 = 2$ and $a_{n+1} = \dfrac{a_n}{2} + \dfrac{1}{a_n}$.
   (a) Prove that $a_n > 0$ for all $n \in \mathbb{N}$.
   (b) Prove[1] that $a_n^2 \geq 2$ for all $n \in \mathbb{N}$.
   (c) Prove[2] that the sequence is decreasing.
   (d) Show that the sequence is convergent.
   (e) Determine[3] what value the sequence converges to.

(4) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and L a real number. Show that $\{a_n\}_{n=1}^{\infty}$ converges to L if and only if both of the subsequences $\{a_{2k}\}_{k=1}^{\infty}$ and $\{a_{2k+1}\}_{k=1}^{\infty}$ converge to L.

[1] Write $a_n - 2$ in terms of a_{n-1} and factor the expression.
[2] Consider $a_{n+1} - a_n$ and use (b).
[3] If the sequence converges to L, explain why $L = \frac{L}{2} + \frac{1}{L}$, and solve for L.