Math 325-002 — Problem Set #6
Due: Thursday, October 6 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences. Either prove[1] or give a counterexample to each of the following:
   (a) If $\{a_n^2\}_{n=1}^{\infty}$ is divergent, then $\{a_n\}_{n=1}^{\infty}$ is divergent.
   (b) If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both diverge, then $\{a_n + b_n\}_{n=1}^{\infty}$ also diverges.
   (c) If $\{a_n\}_{n=1}^{\infty}$ converges and $\{b_n\}_{n=1}^{\infty}$ diverges, then $\{a_n + b_n\}_{n=1}^{\infty}$ diverges.
   (d) Suppose also that $b_n \neq 0$ for all $n \in \mathbb{N}$. If $\{b_n\}_{n=1}^{\infty}$ converges to 0, then $\left\{\dfrac{a_n}{b_n}\right\}_{n=1}^{\infty}$ diverges.

(2) Let $\{a_n\}_{n=1}^{\infty}$ be any bounded sequence (not necessarily convergent) and let $\{b_n\}_{n=1}^{\infty}$ be a sequence that converges to 0. Prove[2] that $\{a_n b_n\}_{n=1}^{\infty}$ converges to 0.

(3) Use the definition to prove that the sequence $\{-\sqrt{n}\}_{n=1}^{\infty}$ diverges to $-\infty$.

(4) Prove that if a sequence $\{a_n\}_{n=1}^{\infty}$ diverges to $-\infty$ then it is not bounded below.

(5) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with $a_n > 0$ for all $n \in \mathbb{N}$. Prove that $\{a_n\}_{n=1}^{\infty}$ converges to 0 if and only if $\left\{\dfrac{1}{a_n}\right\}_{n=1}^{\infty}$ diverges to $+\infty$.

[1] You may use Theorem 10.2 whenever convenient, but make sure you are using something the Theorem says, and nothing it doesn't!

[2] Note that Theorem 10.2 does not apply. Try a different one!