Math 325-002 — Problem Set #4
Due: Thursday, September 22 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Let $\{b_n\}_{n=1}^{\infty}$ be a sequence and L be a real number. Suppose that $b_n = L$ for all $n > 1000000$ (but we know nothing about the first 1000000 values of b_n). Show that $\{b_n\}_{n=1}^{\infty}$ converges to L.

(2) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence, and suppose that $\{a_n\}_{n=1}^{\infty}$ converges to 8.
  (a) Show that there is some real number N such that $a_n \in (5, 11)$ for all natural numbers $n > N$.
  (b) Show that there is some real number N such that $a_n < 8.001$ for all natural numbers $n > N$.
  (c) Explain why $\{n \in \mathbb{N} \mid a_n < 7\}$ is finite (has finitely many elements).

(3) Prove that[1] the sequence $\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ converges to 0.

(4) Let r be a real number. Prove that there is no "next" rational number: i.e., that the set of rational numbers greater than r has no minimum element.

(5) Let S be a set of real numbers, and $T = \{x^2 \mid x \in S\}$.
  (a) Prove that if T is bounded above, then S is bounded above.
  (b) Prove or disprove the converse to part (a).

[1] By $\sqrt{n}$, we mean the positive number whose square is n. Such a number exists by a proof similar to the one that $\sqrt{2}$ exists.