Math 325-002 — Problem Set #2
Due: Thursday, September 8 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Let S be a subset of $\mathbb{R}$ and T be a subset of S. Prove that if S is bounded above then T is also bounded above.

(2) Prove that if S is a subset of $\mathbb{R}$ that is bounded above, then S has infinitely many upper bounds.

(3) Given a subset S of $\mathbb{R}$, a lower bound for S is a real number z such that $z \leq s$ for all $s \in S$. We say S is *bounded below* if S has at least one lower bound. Given a subset S of $\mathbb{R}$, define a new subset $-S$ by

$$-S = \{x \in \mathbb{R} \mid x = -s \text{ for some } s \in S\}.$$

For example, $-\{-2, -1, 1, 3\} = \{-3, -1, 1, 2\}$. Prove that S is bounded below if and only if $-S$ is bounded above.

(4) Suppose S is a subset of $\mathbb{R}$. A real number y is called the *infimum* (also known as greatest lower bound) of S if

- y is a lower bound for S, and
- if z is any lower bound for S then $z \leq y$.

Prove[1] that every nonempty, bounded below subset S of $\mathbb{R}$ has an infimum.

(5) Let S be a subset of $\mathbb{R}$.

(a) Show that the open interval $(1, 2)$ does not have a minimum element.
(b) Show that if y is the minimum of S, then y is the infimum of S.
(c) Show that if ℓ is the infimum of S and $\ell \in S$, then ℓ is the minimum of S.

[1] Hint: The previous problem might be useful.