Math 325-002 — Problem Set #10
Due: Thursday, December 1 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Prove that the function $\sqrt{4-x^2}$ is continuous on the closed interval $[-2,2]$, in the sense of our definition in class, but is not continuous on any open interval that contains $[-2,2]$.

(2) Let $a<b$ be real numbers and $[a,b]$ be a closed interval. Let $\{x_n\}_{n=1}$ be a sequence with $x_n \in [a,b]$ for all n, and assume that $\{x_n\}_{n=1}$ converges to r.
   (a) Prove that $r \in [a,b]$.
   (b) Prove that if f is continuous on the closed interval $[a,b]$, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(r)$.

(3) Use[1] the Intermediate Value Theorem to prove that every positive real number has a square root.

Definition: Let f be a function and r be a real number. We say that f is *differentiable at* r if f is defined at r and the limit
$$\lim_{x\to r}\frac{f(x)-f(r)}{x-r}$$
exists. In this case, we call the limit *the derivative of* f *at* r and write $f'(r)$ for this limit.

(4) Use the definition and any theorems and previous examples of limits to show $f(x)=x^3$ is differentiable at every $r\in\mathbb{R}$ and that $f'(r)=3r^2$.

(5) Use the definition and any theorems and previous examples of limits to show $f(x)=\sqrt{x}$ is differentiable at every $r\in(0,\infty)$ and that $f'(r)=\dfrac{1}{2\sqrt{r}}$.

(6) Show that the function
$$f(x)=\begin{cases}x^2 & \text{if } x\in\mathbb{Q}\\ 0 & \text{if } x\notin\mathbb{Q}\end{cases}$$
is differentiable at $x=0$.

[1] We proved this for the real number 2 and have used this fact without proof for much of the semester; we can give a careful proof now.