Math 325-002 — Problem Set #1
Due: Thursday, September 1 by 7 pm, on Canvas

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) For each of the following sets, which of the properties listed in Proposition 1.2, do *not* hold if one replaces $\mathbb{Q}$ with the indicated set? Give a brief explanation.
   (a) The set of nonnegative integers $\{0, 1, 2, 3, \dots\}$.
   (b) The set of nonnegative rational numbers $\{q \in \mathbb{Q} \mid q \geq 0\}$.
   (c) The set of all integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.

(2) Prove the following "Cancellation of multiplication" property: If x, y, and z are real numbers such that $xy = xz$ and $x \neq 0$, then $y = z$. Your proof should use nothing other than the axioms of the real numbers, just as I did in lecture to show Cancellation of Addition. (You will not need to use the completeness axiom).

(3) Let x and y be real numbers.
   (a) Prove that if x^2 is irrational, then x is irrational.
   (b) Prove that if xy is irrational, then x is irrational or y is irrational.
   (c) Is the converse of (3b) true? Prove or disprove.

(4) Let x be a real number. Use the axioms of $\mathbb{R}$ and facts we have proven in class[1] to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.

(5) Prove that there is no rational number whose square is 3 by mimicking[2] the proof of Theorem 1.1 from class.

(6) Prove[3] that there is no real number whose square is -1.

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[1] Other than this fact itself!

[2] This means many of the steps will be the same, but some details will be different. In particular, "even" and "odd" might not show up in your proof...

[3] Hint: This will require a different idea than problem (5). Instead, you can use without proof that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$, and that $1 > 0$, which we discussed in class.