1. August 23, 2022

What is a number? Certainly the things used to count sheep, money, etc. are numbers: $1, 2, 3, \ldots$. We will call these the *natural numbers* and write $\mathbb{N}$ for the set of all natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}.$$

Since we like to keep track of debts too, we'll allow negatives and 0, which gives us the *integers*:

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}.$$

(The symbol $\mathbb{Z}$ is used since the German word for number is *zahlen*.)

Fractions should count as numbers also, so that we can talk about eating one and two-thirds of a pizza last night. We define a *rational number* to be a number expressible as a quotient of two integers: $\frac{m}{n}$ for $m, n \in \mathbb{Z}$ with $n \neq 0$. For example

$$\frac{5}{3}, \frac{2}{7}, \frac{2019}{2020}$$

are rational numbers. Of course, we often talk about numbers such as "two and a fourth", but that the same as $\frac{9}{4}$. Every integer is a rational number just by taking 1 for the denominator; for example, $7 = \frac{7}{1}$. The set of all rational numbers is written as $\mathbb{Q}$ (for "quotient").

You might not have thought about it before, but an expression of the form $\frac{m}{n}$ is really an "equivalence class": the two numbers $\frac{m}{n}$ and $\frac{a}{b}$ are deemed equal if $mb = na$. For example $\frac{6}{9} = \frac{2}{3}$ because $6 \cdot 3 = 9 \cdot 2$.

We'll talk more about decimals later on, but recall for now that a decimal that terminates is just another way of representing a rational number. For example, 1.9881 is equal to $\frac{19881}{10000}$. Less obvious is the fact that a decimal that repeats also represents a rational number: For example, $1.333\ldots$ is rational (it's equal to $\frac{4}{3}$) and so is $23.91278278278\ldots$. We'll see why this is true later in the semester.

Are these all the numbers there are? Maybe no one in this class would answer "yes", but the ancient Greeks believed for a time that every number was rational. Let's convince ourselves, as the Greeks did eventually, that there must be numbers that are not rational. Imagine a square of side length 1. By the Pythagorean Theorem, the length of its diagonal, call this number c, must satisfy

$$c^2 = 1^2 + 1^2 = 2.$$

That is, there must be a some number whose square is 2 since certainly the length of the diagonal in such a square is representable as a number.

Now, let's convince ourselves that there is no *rational number* with this property. In fact, I'll make this a theorem.

Theorem 1.1. *There is no rational number whose square is* 2.

Preproof Discussion 1. *Before launching a formal proof, let's philosophize about how one shows something does not exist. To show something does not exist, one proves that its existence is not possible. For example, I know that there must not be large clump of plutonium sewn into the mattress of my bed. I know this since, if such a clump existed, I'd be dead by now, and yet here I am, alive and well!*

More generally and formally, one way to prove the falsity of a statement P is to argue that if we assume P to be true then we can deduce from that assumption something that is known to be false. If you can do this, then you have proven P is false. In symbols: If one can prove

$$P \implies \textit{Contradiction}$$

then the statement P must in fact be false.

In the case at hand, letting P be the statement "there is a rational number whose square is 2", the Theorem is asserting that P is false. We will prove this by assuming P is true and deriving an impossibility.

This is known as a proof by contradiction.

Proof. By way of contradiction, assume there were a rational number q such that $q^2 = 2$. By definition of "rational number", we know that q can be written as $\frac{m}{n}$ for some integers m and n such that $n \neq 0$. Moreover, we may assume that we have written q is reduced form so that m and n have no prime factors in common. In particular, we may assume that not both of m and n are even. (If they were both even, then we could simplify the fraction by factoring out common factors of 2's.) Since $q^2 = 2$, $\frac{m^2}{n^2} = 2$ and hence $m^2 = 2n^2$. In particular, this shows m^2 is even and, since the square of an odd number is odd, it must be that m itself is even. So, $m = 2a$ for some integer a. But then $(2a)^2 = 2n^2$ and hence $4a^2 = 2n^2$ whence $2a^2 = n^2$. For the same reason as before, this implies that n must be even. But this contradicts the fact that m and n are not both even.

We have reached a contradiction, and so the assumption that there is a rational number q such that $q^2 = 2$ must be false. □

A version of the previous proof was known even to the ancient Greeks.

Our first major mathematical goal in the class is to make a formal definition of the real numbers. Before we do this, let's record some basic properties of the rational numbers. I'll state this as a Proposition (which is something like a minor version of a Theorem), but we won't prove them; instead, we'll take it for granted to be true based on our own past experience with numbers.

For the rational numbers, we can do arithmetic $(+, -, \times, \div)$ and we also have a notion of size $(<, >)$. The first seven observations below describe the arithmetic, and the last three describe the notion of size.

Proposition 1.2 (Arithmetic and order properties of $\mathbb{Q}$)**.** *The set of rational numbers form an "ordered field". This means that the following ten properties hold:*

(1) *There are operations $+$ and $\cdot$ defined on $\mathbb{Q}$, so that if p, q are in $\mathbb{Q}$, then so are $p + q$ and $p \cdot q$.*
(2) *Each of $+$ and $\cdot$ is a commutative operation (i.e., $p + q = q + p$ and $p \cdot q = q \cdot p$ hold for all rational numbers p and q).*
(3) *Each of $+$ and $\cdot$ is an associative operation (i.e., $(p + q) + r = p + (q + r)$ and $(p \cdot q) \cdot r = p \cdot (q \cdot r)$ hold for all rational numbers p, q, and r).*
(4) *The number 0 is an identity element for addition and the number 1 is an identity element for multiplication. This means that $0 + q = q$ and $1 \cdot q = q$ for all $q \in \mathbb{Q}$.*
(5) *The distributive law holds: $p \cdot (q + r) = p \cdot q + p \cdot r$ for all $p, q, r \in \mathbb{Q}$.*
(6) *Every number has an additive inverse: For any $p \in \mathbb{Q}$, there is a number $-p$ satisfying $p + (-p) = 0$.*
(7) *Every nonzero number has a multiplicative inverse: For any $p \in \mathbb{Q}$ such that $p \neq 0$, there is a number p^{-1} satisfying $p \cdot p^{-1} = 1$.*
(8) *There is a "total ordering" $\leq$ on $\mathbb{Q}$. This means that*
   (a) *For all $p, q \in \mathbb{Q}$, either $p \leq q$ or $q \leq p$.*
   (b) *If $p \leq q$ and $q \leq p$, then $p = q$.*
   (c) *For all $p, q, r \in \mathbb{Q}$, if $p \leq q$ and $q \leq r$, then $p \leq r$.*
(9) *The total ordering $\leq$ is compatible with addition: If $p \leq q$ then $p + r \leq q + r$.*
(10) *The total ordering $\leq$ is compatible with multiplication by non-negative numbers: If $p \leq q$ and $r \geq 0$ then $pr \leq qr$.*

Which of the properties from Proposition 1.2 does $\mathbb{N}$ satisfy?

The commutativity, associativity, distributive law, multiplicative identity, and all of the ordering properties are true for $\mathbb{N}$.

We expect everything from Proposition 1.2 to be true for the real numbers. We will build them into our definition. To define the real numbers $\mathbb{R}$, we take the ten properties listed in the Proposition to be axioms. It turns out the set of real numbers satisfies one key additional property, called the *completeness axiom*, which I cannot state yet.

Axioms. *The set of all real numbers, written $\mathbb{R}$, satisfies the following eleven properties:*

(Axiom 1) *There are operations $+$ and $\cdot$ defined on $\mathbb{R}$, so that if $x, y \in \mathbb{R}$, then so are $x + y$ and $x \cdot y$.*
(Axiom 2) *Each of $+$ and $\cdot$ is a commutative operation.*
(Axiom 3) *Each of $+$ and $\cdot$ is an associative operation.*
(Axiom 4) *The real number 0 is an identity element for addition and the real number 1 is an identity element for multiplication. This means that $0 + x = x$ and $1 \cdot x = x$ for all $x \in \mathbb{R}$.*
(Axiom 5) *The distributive law holds: $x \cdot (y + z) = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$.*
(Axiom 6) *Every real number has an additive inverse: For any $x \in \mathbb{R}$, there is a number $-x$ satisfying $x + (-x) = 0$.*
(Axiom 7) *Every nonzero real number has a multiplicative inverse: For any $x \in \mathbb{R}$ such that $x \neq 0$, there is a real number x^{-1} satisfying $x^{-1} \cdot x = 1$.*
(Axiom 8) *There is a "total ordering" $\leq$ on $\mathbb{R}$. This means that*
  (a) For all $x, y \in \mathbb{R}$, either $x \leq y$ or $y \leq x$.
  (b) If $x \leq y$ and $y \leq z$, then $x \leq z$.
  (c) For all $x, y, z \in \mathbb{R}$, if $x \leq y$ and $y \leq z$, then $x \leq z$.
(Axiom 9) *The total ordering $\leq$ is compatible with addition: If $x \leq y$ then $x + z \leq y + z$ for all z.*
(Axiom 10) *The total ordering $\leq$ is compatible with multiplication by non-negative real numbers: If $x \leq y$ and $z \geq 0$ then $zx \leq zy$.*
(Axiom 11) *The completeness axiom holds. (I will say what this means later.)*

There are many other familiar properties that are consequences of this list of axioms. As an example we can deduce the following property:

> "Cancellation of Addition": For real numbers, $x, y, z \in \mathbb{R}$, if $x + y = z + y$ then $x = z$.

Let's prove this carefully, using just the list of axioms: Assume that $x+y = z+y$. Then we can add $-y$ (which exists by Axiom 6) to both sides to get $(x + y) + (-y) = (z + y) + (-y)$. This can be rewritten as $x + (y + (-y)) = z + (y + (-y))$ (Axiom 3) and hence as $x + 0 = z + 0$ (Axiom 6), which gives $x = z$ (Axiom 4 and Axiom 2).

For another example, we can deduce the following fact from the axioms:

$$r \cdot 0 = 0 \text{ for any real number } r.$$

Let's prove this carefully: Let r be any real number. We have $0+0 = 0$ (Axiom 4) and hence $r\cdot(0+0) = r\cdot 0$. But $r\cdot(0+0) = r\cdot 0+r\cdot 0$ (Axiom 5) and so $r \cdot 0 = r \cdot 0 + r \cdot 0$. We can rewrite this as $0 + r \cdot 0 = r \cdot 0 + r \cdot 0$ (Axiom 4). Now apply the Cancellation of Addition property (which we previously deduced from the axioms) to obtain $0 = r \cdot 0$.

As I said, there are many other familiar properties of the real numbers that follow from these axioms, but I will not list them all. The great news is that all of these familiar properties follow from this short list of axioms. We will prove a couple, but for the most part, I'll rely on your innate knowledge that facts such as $r \cdot 0 = 0$ hold.

2. August 25, 2022

Definition 2.1. A real number is *irrational* if it is not rational.

Making sense of if then statements and quantifier statements.

- The *converse* of the statement "If P then Q" is the statement "If Q then P".
- The *contrapositive* of the statement "If P then Q" is the statement "If not Q then not P".
- Any if then statement is equivalent to its contrapositive, but not necessarily to its converse!

(1) For each of the following statements, write its contrapositive and its converse. Is the original/contrapositive/converse true or false for real numbers a, b? Explain why (but don't prove).
   (a) If a is irrational, then $1/a$ is irrational.
   (b) If a and b are irrational, then ab is irrational.
   (c) If $a > 3$, then $a^2 > 9$.

(1) true; contrapositive is "If $1/a$ is rational, a is rational" is true; converse is "if $1/a$ is irrational, then a is irrational" is true.
(2) false; contrapositive is "If ab is rational, either a or b is rational" is false; converse is "if ab is irrational then a and b are irrational" is false.
(3) true; contrapositive is "if $a^2 \leq 9$, then $a \leq 3$ is true; converse is "if $a^2 > 9$ then $a > 3$" is false.

- The symbol for "for all" is $\forall$ and the symbol for there exists is $\exists$.
- The negation of "For all $x \in S$, P" is "There exists $x \in S$ such that not P".
- The negation of "There exists $x \in S$ such that P" is "For all $x \in S$, not P".

(2) Rewrite each statement with symbols in place of quantifiers, and write its negation. Is the original statement true or false? Explain why (but don't prove them).
 (a) There exists $x \in \mathbb{Q}$ such that $x^2 = 2$.
 (b) For all $x \in \mathbb{R}$, $x^2 > 0$.
 (c) For all $x \in \mathbb{R}$ such that[1] $x \neq 0$, $x^2 > 0$.
 (d) For all $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x < y$.
 (e) There exists $x \in \mathbb{R}$ such that for all $y \in \mathbb{R}$, $x < y$.

(1) $\exists x \in \mathbb{Q} : x^2 = 2$ is false. Negation: $\forall x \in \mathbb{Q}, x^2 \neq 2$.
(2) $\forall x \in \mathbb{R}$, $x^2 > 0$ is false. Negation: $\exists x \in \mathbb{R} : x^2 \leq 0$.
(3) $\forall x \in \mathbb{R} : x \neq 0, x^2 > 0$ is true. Negation: $\exists x \in \mathbb{R} : x \neq 0, x^2 \leq 0$.
(4) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x < y$ is true. Negation: $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x \geq y$.
(5) $\exists x \in \mathbb{R} : \forall y \in \mathbb{R}, x < y$ is false. Negation: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R} : x \geq y$.

Proving if then statements and quantifier statements.

- The general outline of a direct proof of "If P then Q" goes
  (1) Assume P.
  (2) Do some stuff.
  (3) Conclude Q.
- Often it is easier to prove the contrapositive of an if then statement than the original, especially when the negation of the hypothesis or conclusion is something negative.
- The general outline of a proof of "For all $x \in S$, P" goes
  (1) Let $x \in S$ be arbitrary.
  (2) Do some stuff.
  (3) Conclude that P holds for x.
- To prove a there exists statement, you just need to give an example. To prove "There exists $x \in S$ such that P" directly:
  (1) Consider[2] $x =$[some specific element of S].
  (2) Do some stuff.
  (3) Conclude that P holds for x.

[1] In a statement of the form "For all $x \in S$ such that Q, P", the "such that Q" part is part of the hypothesis: it is restricting the set S that we are "alling"' over.

(3) Let x and y be real numbers. Use the axioms of $\mathbb{R}$ to prove[3] that $x \geq y$ if and only if $-y \geq -x$.
(4) Let x be a real number. Show that if x^2 is irrational, then x is irrational.
(5) Let x be a real number. Use the axioms of $\mathbb{R}$ and facts we have proven in class to show that if there exists a real number y such that $xy = 1$, then $x \neq 0$.
(6) Prove that[4] for all $x \in \mathbb{R}$ such that $x \neq 0$, we have $x^2 \neq 0$.
(7) Prove that there exists some $x \in \mathbb{R}$ such that for every $y \in \mathbb{R}$, $xy = x$.
(8) Prove[5] that (2d) is true and (2e) is false.
(9) Let $S \subseteq \mathbb{R}$ be a set of real numbers. Apply your results above to prove that if for every $x \in S$, x^2 is irrational, then for every $y \in S$, y is irrational.
(10) Prove that $1 > 0$.
(11) Let x, y be real numbers. Prove that if $x \leq 0$ and $y \leq 0$, then $xy \geq 0$.

(3) Let $x \geq y$. Adding $(-x)+(-y)$ to both sides (which exists by Axiom 6), we obtain $-y = x + ((-x) + (-y)) \geq y + ((-x)+(-y)) = -x$ (by Axiom 9 and Axiom 5). Conversely, let $-x \leq -y$. Adding $x + y$ to both sides, we obtain $y = (x + y) + (-x) \leq (x + y) + (-y) = x$ (by Axiom 9 and Axiom 5).
(6) Let x and y be nonzero real numbers. By Axiom 7, there are element $x^{-1}, y^{-1} \in \mathbb{R}$ such that $xx^{-1} = 1$ and $yy^{-1} = 1$. Then $xy \cdot (x^{-1}y^{-1}) = (xx^{-1})(yy^{-1}) = 1$, using Axioms 2 and 3 in the first equality and Axiom 5 in the second. By the previous fact (applies to xy) we conclude that $xy \neq 0$.
(7) Consider $x = 1$. Let $y \in \mathbb{R}$. By Axiom 4, we have $xy = 1y = y$. Thus, for all $y \in \mathbb{R}$, we have $xy = x$.
(8) (2d): Let $x \in \mathbb{R}$. Consider $y = x + 1$. Since $1 > 0$ we have $y = x + 1 > x + 0 = x$. Thus, for each $x \in \mathbb{R}$, we have some y such that $x < y$. (2e): We claim this is false. Suppose, for the sake of contradiction that this was true, and let x

[2]How you found this x is logically irrelevant to an existence proof, and should not be included.
[3]Hint: You may want to add something to both sides.
[4]Hint: Use (5).
[5]You can "work out of order here" and *use* (10) now.

be as in the statement. Then for any $y \in \mathbb{R}$, we have $x < y$. But, for $y = x$, the inequality $x < y$ is false. This is a contradiction, so the statement must be false.

(10) First we establish two lemmas.

Lemma: For real numbers $x \in \mathbb{R}$ we have $-(-x) = x$.

Proof: We have

$$(-x) + (-(-x)) = 0$$

so

$$\begin{aligned} -(-x) &= 0 + -(-x) = (x + (-x)) + (-(-x)) \\ &= x + ((-x) + (-(-x))) = x. \quad \square \end{aligned}$$

Lemma: For real numbers $x, y \in \mathbb{R}$ we have $(-x)y = -(xy)$.

Proof: We have that

$$0 = 0y = (x + (-x))y = xy + (-x)y.$$

Adding $-(xy)$ to both sides we get

$$\begin{aligned} -(xy) &= -(xy) + (xy + (-x)y) \\ &= (-(xy) + (-x)y) + (-x)y \\ &= 0 + (-x)y = (-x)y. \quad \square \end{aligned}$$

We proceed with the proof. We either have $1 \geq 0$ or $1 \leq 0$. Suppose that $1 \leq 0$. Then $-1 \geq 0$, so

$$(-1)(-1) \geq (-1)0 = 0.$$

But

$$(-1)(-1) = -(1(-1)) = -(-1) = 1,$$

so $1 \geq 0$, contradicting the hypothesis.

3. August 30, 2022

I owe you a statement of the very important Completeness Axiom. Before we get there, I want to recall an axiom of $\mathbb{N}$ that we haven't discussed yet. It pertains to minimum elements in sets. Let's be precise and define minimum element.

Definition 3.1. Let S be a set of real numbers. A *minimum* element of S is a real number x such that

(1) $x \in S$, and
(2) for all $y \in S$, $x \leq y$.

In this case, we write $x = \min(S)$.

The definition of *maximum* is the same except with the opposite inequality.

Axiom 3.2 (Well-ordering axiom)**.** *Every nonempty subset of* $\mathbb{N}$ *has a minimum element.*

Example 3.3. If S is the set of even multiples of 7, then S has 14 as its minimum.

We generally like to say *the* minimum, rather than *a* minimum. To justify this, let's prove the following.

Proposition 3.4. *Let S be a set of real numbers. If S has a minimum, then the minimum is unique.*

Preproof Discussion 2. *The proposition has the general form "If a thing with property P exists, then it is unique".*

How do we prove a statement such as "If a thing with property P exists, then it is unique"? We argue that if two things x and y both have property P, then x and y must be the same thing.

Proof of Proposition 3.4. Let S be a set of real numbers, and let x and y be two minima of S. Applying part (1) of the definition of minimum to y, we have $y \in S$. Applying part (2) of the definition of minimum to x and the fact that $y \in S$, we get that $x \leq y$. Switching roles, we get that $y \leq x$. Thus $x = y$.

We conclude that if a minimum exists, it is necessarily unique. $\square$

The previous proposition plus the Well-Ordering Axiom together imply that every nonempty subset of $\mathbb{N}$ has exactly one minimum element. A similar proof shows that if a maximum exists, it is necessarily unique. Could a set fail to have a maximum or a minimum? Yes!

Example 3.5.

(1) The empty set $\varnothing$ has no minimum and no maximum element. (There is no $s \in \varnothing$!)

(2) The set of natural numbers $\mathbb{N}$ has 1 as a minimum, but has no maximum. (Suppose there was: if $n = \max(\mathbb{N})$ was the maximum, then $n < n + 1 \in \mathbb{N}$ gives a contradiction.)

(3) The open interval $(0, 1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$ has no minimum and no maximum. (Exercise later.)

Definition 3.6. Let S be any subset of $\mathbb{R}$. A real number b is called an *upper bound* of S provided that for every $s \in S$, we have $s \leq b$.

For example, the number 1 is an upper bound for the interval $(0, 1)$. The number 182 is also an upper bound of this set and so is π. It is pretty clear that 1 is the "best" (i.e., smallest) upper bound for this set, in the sense that every other upper bound of $(0,1)$ must be at least as big as 1. Let's make this official:

Proposition 3.7. *If b is an upper bound of the set $(0,1)$, then $b \geq 1$.*

I will prove this claim using just the axioms of the real numbers (in fact, I will only use the first 10 axioms):

Proof. Suppose b is an upper bound of the set $(0,1)$. By way of contradiction, suppose $b < 1$. (Our goal is to derive a contradiction from this.)

Consider the number $y = \frac{b+1}{2}$ (the average of b and 1). I will argue that $b < y$ and $b \geq y$, which is not possible.

Since we are assuming $b < 1$, we have $\frac{b}{2} < \frac{1}{2}$ and hence

$$b = \frac{2b}{2} = \frac{b}{2} + \frac{b}{2} < \frac{b}{2} + \frac{1}{2} = \frac{b+1}{2} = y.$$

So, $b < y$.

Similarly,

$$1 = \frac{1+1}{2} > \frac{b+1}{2} = y$$

so that

$$y < 1.$$

Since $\frac{1}{2} \in S$ and b is an upper bound of S, we have $\frac{1}{2} \leq b$. Since we already know that $b < y$, it follows that $\frac{1}{2} < y$ and hence $0 < y$. We have proven that $y \in (0,1)$. But, remember that b is an upper bound of $(0,1)$, and so we get $y \leq b$ by definition.

To summarize: given an upper bound b of $(0,1)$, starting with the assumption that $b < 1$, we have deduced the existence of a number y such that both $b < y$ and $y \leq b$ hold. As this is not possible, it must be that $b < 1$ is false, and hence $b \geq 1$. □

This claim proves the (intuitively obvious) fact that 1 is "least upper bound" of the set $(0,1)$. The notion of "least upper bound" will be an extremely important one in this class.

Definition 3.8. A subset S of $\mathbb{R}$ is called *bounded above* if there exists at least one upper bound for S. That is, S is bounded above provided there is a real number b such that for all $s \in S$ we have $s \leq b$.

For example, $(0,1)$ is bounded above, by for example 50.

The subset $\mathbb{N}$ of $\mathbb{R}$ is not bounded above — there is no real number that is larger than every natural number. This fact is surprisingly non-trivial to deduce just using the axioms; in fact, one needs the

Completeness Axiom to show it. But of course our intuition tells us that it is obviously true.

Let's give a more interesting example of a subset of $\mathbb{R}$ that is bounded above.

Example 3.9. Define S to be those real numbers whose squares are less than 2:

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

I claim S is bounded above. In fact, I'll prove 2 is an upper bound: Suppose $x \in S$. If $x > 2$, then $x \cdot x > x \cdot 2$ and $x \cdot 2 > 2 \cdot 2$, and hence $x^2 > 4 > 2$. This contradicts the fact that $x \in S$. So, we must have $x \leq 2$.

A nearly identical argument shows that 1.5 is also an upper bound (since $1.5^2 = 2.25 > 2$) and similarly one can show 1.42 is an upper bound. But 1.41 is not an upper bound. For note that $1.411^2 = 1.99091$ and so $1.41 \in S$ but $1.411 > 1.41$.

Question: What is the smallest (or least) upper bound for this set S? Clearly, it ought to be $\sqrt{2}$ (i.e., the positive number whose square is equal to exactly 2), but there's a catch: how do we know that such real number exists?

Definition 3.10. Suppose S is subset of $\mathbb{R}$ that is bounded above. A *supremum* (also known as a *least upper bound*) of S is a number ℓ such that

(1) ℓ is an upper bound of S (i.e., $s \leq \ell$ for all $s \in S$) and
(2) if b is any upper bound of S, then $\ell \leq b$.

In this case we write $\sup(S) = \ell$.

Example 3.11. 1 is a supremum of $(0, 1)$. Indeed, it is clearly an upper bound, and in the "Claim" above, we proved that if b is any upper bound of $(0, 1)$ then $b \geq 1$. Note that this example shows that a supremum of S does not necessarily belong to S.

Example 3.12. I claim 1 is a supremum of

$$(0, 1] = \{x \in \mathbb{R} \mid 0 < x \leq 1\}.$$

It is by definition an upper bound. If b is any upper bound of $(0, 1]$ then, since $1 \in (0, 1]$, by definition we have $1 \leq b$. So 1 is the supremum of $(0, 1]$.

Observation 3.13. *Let S be a set of real numbers. Suppose that $b \in S$ and that b is an upper bound for S. Then*

(1) b is the maximum of S, and
(2) b is a supremum of S.

The subset $\mathbb{N}$ does not have a supremum since, indeed, it does not have any upper bounds at all.

Can you think of an example of a set that is bounded above but has no supremum? There is only one such example and it is rather silly: the empty set is bounded above. Indeed, every real number is an upper bound for the empty set. So, there is no least upper bound.

Having explained the meaning of the term "supremum", I can finally state the all-important completeness axiom:

Axiom (Completeness Axiom)**.** *Every nonempty, bounded-above subset of* $\mathbb{R}$ *has a supremum.*

4. September 1, 2022

(1) Write, in simplified form, the negation of the statement "b is an upper bound for S".

There exists some $x \in S$ such that $x > b$.

(2) Write, in simplified form, the negation of the statement "S is bounded above".

For every $b \in \mathbb{R}$, there exists $x \in S$ such that $x > b$.

(3) Let S be a set of real numbers and suppose that $\ell = \sup(S)$.
   (a) If $x > \ell$, what is the most concrete thing you can say about x and S?
   (b) If $x < \ell$, what is the most concrete thing you can say about x and S?

(a) $x \notin S$. (b) There exists some $y \in S$ such that $y > x$.

(4) Let S be a set of real numbers, and let $T = \{2s \mid s \in S\}$. Prove that if S is bounded above, then T is bounded above.

Assume that S is bounded above. Then there is some upper bound b for S, so for every $s \in S$, we have $b \geq s$. We claim that $2b$ is an upper bound for T. Indeed, if $t \in T$, then we can write $t = 2s$ for some $s \in S$, and $s \leq b$ implies $t = 2s \leq 2b$. Thus, T is bounded above.

(5) Let S be a set of real numbers. Show that if S has a supremum, then it is unique.

Suppose both x and y are both suprema of the same subset S of $\mathbb{R}$. Then, since y is an upper bound of S and x is a supremum of S, by part (2) of the definition of "supremum" we have $y \geq x$. Likewise, since x is an upper bound of S and y is a supremum of S, we have $x \geq y$ by definition. Since $x \leq y$ and $y \leq x$, we conclude $x = y$.

(6) Let S be a set of real numbers, and let $T = \left\{ \frac{s}{2} \mid s \in S \right\}$. Directly prove that if S is unbounded above, then T is unbounded above.

Assume that S is unbounded above. To show that T is unbounded above, let b be a real number. Since S is unbounded above, $2b$ is not an upper bound for S, so there is some $s \in S$ with $s > 2b$. Then $\frac{s}{2} > b$. By definition of T, we have $\frac{s}{2} \in T$, so b is not an upper bound of T. We conclude that T is unbounded above.

5. September 6, 2022

Let us now explore consequences of the completeness axiom. We know that there is no rational number whose square is 2; now we show that there is indeed a real number whose square is two.

Proposition 5.1. *There is a positive real number whose square is* 2*.*

Proof. Define S to be the subset

$$S = \{x \in \mathbb{R} \mid x^2 < 2\}.$$

S is nonempty since, for example, $1 \in S$, and it is bounded above, since, for example, 2 is an upper bound for S, as we showed earlier. So, by the Completeness Axiom, S has a least upper bound, and we know it is unique from the proposition above. Let us call it ℓ. I will prove $\ell^2 = 2$.

We know one of $\ell^2 > 2$, $\ell^2 < 2$ or $\ell^2 = 2$ must hold. We prove $\ell^2 = 2$ by showing that both $\ell^2 > 2$ and $\ell^2 < 2$ are impossible.

We start by observing that $1 \leq \ell \leq 2$. The inequality $1 \leq \ell$ holds since $1 \in S$ and ℓ is an upper bound of S, and the inequality $\ell \leq 2$ holds since 2 is an upper bound of S and ℓ is the least upper bound of S.

Suppose $\ell^2 < 2$. We show this leads to a contradiction by showing that ℓ is not an upper bound of S in this case. We will do this by constructing a number that is ever so slightly bigger than ℓ and belongs to S. Let $\varepsilon = 2 - \ell^2$. Then $0 < \varepsilon \leq 1$ (since $\ell^2 < 2$ and $\ell^2 \geq 1$). We will now show that $\ell + \varepsilon/5$ is in S: We have

$$(\ell + \varepsilon/5)^2 = \ell^2 + \frac{2}{5}\ell\varepsilon + \frac{\varepsilon^2}{25} = \ell^2 + \varepsilon(\frac{2\ell}{5} + \frac{\varepsilon}{25}).$$

Now, using $\ell \leq 2$ and $0 < \varepsilon \leq 1$, we deduce

$$0 < \frac{2\ell}{5} + \frac{\varepsilon}{25} \leq \frac{4}{5} + \frac{\varepsilon}{25} < 1.$$

Putting these equations and inequalities together yields

$$(\ell + \frac{\varepsilon}{5})^2 < \ell^2 + \varepsilon = 2.$$

So, $\ell + \frac{\varepsilon}{5} \in S$ and yet $\ell + \frac{\varepsilon}{5} > \ell$, contradicting the fact that l is an upper bound of S. We conclude $\ell^2 < 2$ is not possible.

Assume now that $\ell^2 > 2$. Our strategy will be to construct a number ever so slightly smaller than ℓ, which therefore cannot be an upper bound of S, and use this to arrive at a contradiction. Let $\delta = \ell^2 - 2$. Then $0 < \delta \leq 2$ (since $\ell \leq 2$ and hence $\ell^2 - 2 \leq 2$). Since $\delta > 0$, we have $\ell - \frac{\delta}{5} < \ell$. Since ℓ is the least upper bound of S, $\ell - \frac{\delta}{5}$ must not be an upper bound of S. By definition, this means that there is $r \in S$ such that $\ell - \frac{\delta}{5} < r$. Since $\delta \leq 2$ and $\ell \geq 1$, it follows that $\ell - \frac{\delta}{5}$ is positive and hence so is r. We may thus square both sides of $\ell - \frac{\delta}{5} < r$ to obtain

$$(\ell - \frac{\delta}{5})^2 < r^2.$$

Now

$$(\ell - \frac{\delta}{5})^2 = \ell^2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = \delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25}$$

since $\ell^2 = \delta + 2$. Moreover,

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} = 2 + \delta(1 - \frac{2\ell}{5} + \frac{\delta}{25}) \geq 2 + \delta(1 - \frac{4}{5} + \frac{\delta}{25})$$

since $\ell \leq 2$. We deduce that

$$\delta + 2 - \frac{2\ell\delta}{5} + \frac{\delta^2}{25} \geq 2 + \delta(\frac{1}{5}) \geq 2.$$

Putting these inequalities together gives $r^2 > 2$, contrary to the fact that $r \in S$. We conclude that $\ell^2 > 2$ is also not possible.

Since $\ell^2 < 2$ and $\ell^2 > 2$ are impossible, we must have $\ell^2 = 2$. $\square$

The collection of rational numbers does not satisfy the completeness axiom and indeed it is precisely the completeness axiom that differentiates $\mathbb{R}$ from $\mathbb{Q}$.

Example 5.2. Within the set $\mathbb{Q}$ the subset $S = \{x \in \mathbb{Q} \mid x^2 < 2\}$ does not have a supremum. That is, no matter which rational number you pick that is an upper bound for S, you may always find an even smaller one that is also an upper bound of S.

It is precisely the completeness axiom that assures us that everything that ought to be a number (like the length of the diagonal of a square with side length 1) really is a number. It gives us that there are "no holes" in the real number line — the real numbers are *complete.*

For example, we can use it to prove that $\sqrt[8]{147}$ exists: Let $S = \{x \in \mathbb{R} \mid x^8 < 147\}$. Then S is nonempty (e.g., $0 \in S$) and bounded above (e.g., 50 is an upper bound) and so it must have a supremum ℓ. A proof similar to (but even messier than) the proof of Proposition 5.1 above shows that ℓ satisfies $\ell^8 = 147$.

The completeness axiom is also at the core of the Intermediate Value Theorem and many of the other major theorems we will cover in this class.

We also need the completeness axiom to understand the relationship between $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$.

Theorem 5.3. *If x is any real number, then there exists a natural number n such that $n > x$.*

This looks really stupid at first. How could it be false? But consider: there are examples of ordered fields, i.e. situations in which Axioms 1–10 hold, in which this Theorem is not true! So, its proof must rely on the Completeness Axiom.

Proof. Let x be any real number. By way of contradiction, suppose there is no natural number n such that $n > x$. That is, suppose that for all $n \in \mathbb{N}$, $n \leq x$. Then $\mathbb{N}$ is a bounded above (by x). Since it is also clearly nonempty, by the Completeness Axiom, $\mathbb{N}$ has a supremum, call it ℓ. Consider the number $y := \ell - 1$. Since $y < \ell$ and ℓ is the supremum of $\mathbb{N}$, y cannot be an upper bound of $\mathbb{N}$. So, there must be some $m \in \mathbb{N}$ such that such that $\ell - 1 < m$. But then by adding 1 to both sides of this inequality we get $\ell < m + 1$ and, since $m + 1 \in \mathbb{N}$, this contradicts that assumption that ℓ is the supremum of $\mathbb{N}$.

We conclude that, given any real number x, there must exist a natural number n such that $n > x$. $\square$

Corollary 5.4 (Archimedean Principle)**.** *If $a \in \mathbb{R}$, $a > 0$, and $b \in \mathbb{R}$, then for some natural number n we have $na > b$.*

"No matter how small a is and how large b is, if we add a to itself enough times, we can overtake b."

Proof. We apply Theorem 5.3 to the real number $x = \frac{b}{a}$. It gives that there is a natural number n such that $n > x = \frac{b}{a}$. Since $a > 0$, upon multiplying both sides by a we get $n \cdot a > b$. □

Theorem 5.5 (Density of the Rational Numbers). *Between any two distinct real numbers there is a rational number; more precisely, if $x, y \in \mathbb{R}$ and $x < y$, then there exists $q \in \mathbb{Q}$ such that $x < q < y$.*

Proof. We will prove this by consider two cases: $x \geq 0$ and $x < 0$.

Let us first assume $x \geq 0$. We apply the Archimedean Principle using $a = y - x$ and $b = 1$. (The Principle applies as $a > 0$ since $y > x$.) This gives us that there is a natural number $n \in \mathbb{N}$ such that

$$n \cdot (y - x) > 1$$

and thus

$$0 < \frac{1}{n} < y - x.$$

Consider the set $S = \{p \in \mathbb{N} \mid p\frac{1}{n} > x\}$. Since $\frac{1}{n} > 0$, using the Archimedean principle again, there is at least one natural number $p \in S$. By the Well Ordering Axiom, there is a smallest natural number $m \in S$.

We claim that $\frac{m-1}{n} \leq x$. Indeed, if $m > 1$, then $m - 1 \in \mathbb{N} \setminus S$ (because $m - 1$ is less than the minimum), so $\frac{m-1}{n} \leq x$; if $m = 1$, then $m - 1 = 0$, so $\frac{m-1}{n} = 0 \leq x$.

So, we have

$$\frac{m-1}{n} \leq x < \frac{m}{n}$$

By adding $\frac{1}{n}$ to both sides of $\frac{m-1}{n} \leq x$ and using that $\frac{1}{n} < y - x$, we get

$$\frac{m}{n} \leq x + \frac{1}{n} < x + (y - x) = y$$

and hence

$$x < \frac{m}{n} < y.$$

Since $\frac{m}{n}$ is clearly a rational number, this proves the result in this case (when $x > 0$).

We now consider the case $x < 0$. The idea here is to simply "shift" up to the case we've already proven. By Theorem 5.3, we can find a natural number j such that $j > -x$ and thus $0 < x + j < y + j$. Using the first case, which we have already proven, applied to the number $x + j$ (which is positive), there is a rational number q such that

$x + j < q < y + j$. We deduce that $x < q - j < y$, and, since $q - j$ is also rational, this proves the theorem in this case. □

6. September 8, 2022

(1) Let W be the set of real numbers x that satisfy the inequality $x^3 + x < 10$.
 (a) Write W mathematically in set notation.
 (b) Does W have a supremum? Why or why not?
 (c) Is $\sup(W) = 1$? Why or why not?
 (d) Is $\sup(W) = 4$? Why or why not?

> (a) $W = \{x \in \mathbb{R} \mid x^3 + x < 10\}$.
> (b) Yes. It is nonempty, since $0 \in W$, and bounded above, e.g., by 3: if $x > 3$, then $x^3 + x > 3^3 + 3 = 30$, so $x \notin W$.
> (c) No: 1 is not an upper bound, because $1.5 \in W$.
> (d) No: 3 is an upper bound, and $3 < 4$.

(2) Use the Archimedean Principle to show that for any positive number $\varepsilon > 0$, there is a natural number n such that $0 < \varepsilon < \frac{1}{n}$.

(3) Prove that the supremum of the set $S = \left\{1 - \frac{1}{n} \mid n \in \mathbb{N}\right\}$ is 1.

(4) Let S be a set of real numbers, and suppose that $\sup(S) = \ell$. Let $T = \{s + 7 \mid s \in S\}$. Prove that $\sup(T) = \ell + 7$.

> First, we show that $\ell + 7$ is an upper bound of T. Let $t \in T$. Then there is some $s \in S$ such that $t = s + 7$. Since $s \leq \ell$, we have $t = s + 7 < \ell + 7$, so $\ell + 7$ is indeed an upper bound. Next, let b be an upper bound for T. We claim that $b - 7$ is an upper bound for S. Indeed, if $s \in S$, then $s + 7 \in T$ so $s + 7 \leq b$, so $s \leq b - 7$. Then, by definition of supremum, we have $b - 7 \geq \ell$, eso $b \geq \ell + 7$.

(5) Prove the following:

Corollary 6.1 (Density of irrational numbers)**.** *For any real numbers x, y with $x < y$, there is some irrational number z such that $x < z < y$.*

> Let $x < y$ be real numbers. Then we have $x - \sqrt{2} < y - \sqrt{2}$. By density of rationals, there is some rational number q such that $x - \sqrt{2} < q < y - \sqrt{2}$. Then $x < q + \sqrt{2} < y$. Since q is rational and $\sqrt{2}$ is irrational, $z = q + \sqrt{2}$ is irrational, and hence the number we seek.

(6) True or false & justify: There is a rational number x such that $|x^2 - 2| = 0$.

> False: this would imply that x is a rational number whose square is 2.

(7) True or false & justify: There is a rational number x such that $|x^2 - 2| < \frac{1}{1000000}$.

> True: By density of rational numbers, there is a rational number q such that $\sqrt{2} - \frac{1}{5000000} < q < \sqrt{2}$. Then
>
> $$\begin{aligned} |x^2 - 2| &= |x - \sqrt{2}|\,|x + \sqrt{2}| \\ &< \frac{1}{5000000}\left(\frac{1}{5000000} + 4\right) \\ &< \frac{1}{5000000} \cdot 5 \\ &= \frac{1}{1000000}. \end{aligned}$$

7. September 13, 2022

We now turn our attention to the next major topic of this class: sequences of real numbers. We will spend the next few weeks developing their properties carefully and rigorously. Sequences form the foundation for much of what we will cover for the rest of the semester.

Definition 7.1. A *sequence* is an infinite list of real numbers indexed by $\mathbb{N}$:

$$a_1, a_2, a_3, \ldots.$$

(Equivalently, a sequence is a function from $\mathbb{N}$ to $\mathbb{R}$: the value of the function at $n \in \mathbb{N}$ is written as a_n.)

We will usually write $\{a_n\}_{n=1}^{\infty}$ for a sequence.

Example 7.2. To describe sequences, we will typically give a formula for the n-th term, a_n, either an explicit one or a recursive one. On rare occasion we'll just list enough terms to make the pattern clear. Here are some examples:

(1) $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ is the sequence that starts

$$4, \frac{11}{2}, \frac{14}{3}, \frac{21}{4}, \frac{24}{5}, \ldots.$$

(2) Let $\{a_n\}_{n=1}^{\infty}$ be defined by $a_1 = 1, a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$. This gives the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

This is an example of a recursively defined sequence. It is the famed *Fibonacci sequence*.

(3) Let $\{c_n\}_{n=1}^{\infty}$ be the sequence whose n-th term is the n-th smallest positive prime integer:

$$2, 3, 5, 7, 11, 13, 17, 19, 23, \ldots.$$

Note that here I have not really given an explicit formula for the terms of the sequence, but it is possible to describe an algorithm that lists every term of the sequence in order.

You have all probably seen an "intuitive" definition of the limit of a sequence before. For example, you probably believe that

$$5 + (-1)^n \frac{1}{n}$$

converges to 5. Let's give the rigorous definition.

Definition 7.3. Let $\{a_n\}_{n=1}^{\infty}$ be an arbitrary sequence and L a real number. We say $\{a_n\}_{n=1}^{\infty}$ *converges* to L provided the following condition is met:

> For every real number $\varepsilon > 0$, there is a real number N such that $|a_n - L| < \varepsilon$ for all natural numbers n such that $n > N$.

This is an extremely important definition for this class. Learn it by heart!

In symbols, the definition is

> A sequence $\{a_n\}_{n=1}^{\infty}$ converges to L provided
> $\forall \varepsilon > 0, \exists N \in \mathbb{R} : \forall n \in \mathbb{N}$ s.t. $n > N$, $|a_n - L| < \varepsilon$.

It's a complicated definition — three quantifiers!

Here is what the definition is saying somewhat loosely: No matter how small a number ε you pick, so long as it is positive, if you go far enough out in the sequence, all of the terms from that point on will be within a distance of ε of the limiting value L.

Example 7.4. To say that the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 5 + (-1)^n \frac{1}{n}$ converges to 5 gives us a different statement for every $\varepsilon > 0$. For example:

- Setting $\varepsilon = 3$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < 3$. Namely, we can take $N = 0$, since for *every* term a_n of the sequence, $|a_n - 5| < 3$ holds true.
- Setting $\varepsilon = \frac{1}{3}$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < \frac{1}{3}$. We cannot take $N = 0$ anymore, since $1 > 0$ and $|a_1 - 5| = 1 > \frac{1}{3}$. However, we can take $N = 3$, since for $n > 3$, $|a_n - 5| = \frac{1}{n} < \frac{1}{3}$.
- Setting $\varepsilon = 1/1000000$, there is a number N such that for every natural number $n > N$, $|a_n - 5| < 1/1000000$. We need a bigger N; now $N = 1000000$ works.

In general, our choice of N may depend on ε, which is OK since our definition is of the form $\forall \varepsilon > 0, \exists N \ldots$ rather than $\exists N : \forall \varepsilon > 0 \ldots$.

Example 7.5. I claim the sequence $\{a_n\}_{n=1}^{\infty}$ where $a_n = 5 + (-1)^n \frac{1}{n}$ converges to 5. I'll give a rigorous proof, along with some commentary and "scratch work" within the parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: Given this ε, our goal is to find N so that if $n > N$, then $|5 + (-1)^n \frac{1}{n} - 5| < \varepsilon$. The latter simplifies to $\frac{1}{n} < \varepsilon$, which in turn is equivalent to $\frac{1}{\varepsilon} < n$ since ε and n are both positive. So, it seems we've found the N that "works". Back to the formal proof....)

Let $N = \frac{1}{\varepsilon}$. Then $\frac{1}{N} = \varepsilon$, since ε is positive.

(Comment: We next show that this is the N that "works" in the definition. Since this involves proving something about every natural number that is bigger than N, we start by picking one.)

Pick any $n \in \mathbb{N}$ such that $n > N$. Then $\frac{1}{n} < \frac{1}{N}$ and hence

$$|a_n - 5| = |5 + (-1)^n \frac{1}{n} - 5| = |(-1)^n \frac{1}{n}| = \frac{1}{n} < \frac{1}{N} = \varepsilon.$$

This proves that $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ converges to 5. $\square$

Remark 7.6. A direct proof that a certain sequence converges to a certain number follows the general outline:

- Let $\varepsilon > 0$ be given. (or, if your prefer, "Pick $\varepsilon > 0$.")
- Let $N =$ [expression in terms of ε from scratch work].
- Let $n \in \mathbb{N}$ be such that $n > N$.

- [Argument that $|a_n - L| < \varepsilon$.]
- Thus $\{a_n\}_{n=1}^{\infty}$ converges to L.

Example 7.7. I claim that the sequence

$$\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$$

converges to $\frac{2}{5}$. Again I'll give a proof with commentary and scratch work in parentheses.

Proof. Let $\varepsilon > 0$ be given.

(Scratch work: We need n to be large enough so that

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This simplifies to $\left|\frac{-7}{25n+5}\right| < \varepsilon$ and thus to $\frac{7}{25n+5} < \varepsilon$, which we can rewritten as $\frac{7}{25\varepsilon} - \frac{1}{5} < n$.)

Let $N = \frac{7}{25\varepsilon} - \frac{1}{5}$. We solve this equation for ε: We get $\frac{7}{25\varepsilon} = \frac{5N+1}{5}$ and hence $\frac{25\varepsilon}{7} = \frac{5}{5N+1}$, which gives finally

$$\varepsilon = \frac{7}{25N+5}.$$

(Next we show this value of N works....)

Now pick any $n \in \mathbb{N}$ is such that $n > N$. Then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| = \left|\frac{10n-5-10n-2}{25n+5}\right| = \frac{7}{25n+5}.$$

Since $n > N$, $25n+5 > 25N+5$ and hence

$$\frac{7}{25n+5} < \frac{7}{25N+5} = \varepsilon.$$

We have proven that if $n \in \mathbb{N}$ and $n > N$, then

$$\left|\frac{2n-1}{5n+1} - \frac{2}{5}\right| < \varepsilon.$$

This proves $\left\{\frac{2n-1}{5n+1}\right\}_{n=1}^{\infty}$ converges to $\frac{2}{5}$. □

8. September 15, 2022

(1) Let c be a real number. Prove that the constant sequence $\{c\}_{n=1}^{\infty}$ converges to c.

> Let $\varepsilon > 0$. Take $N = 0$ (or $N = 588$, or $N = -10000000$, or any other real number). For any natural number $n > N$, we have $|a_n - c| = 0 < \varepsilon$. Thus the sequence converges to c.

(2) Prove that the sequence $\left\{\frac{1}{\sqrt{n}}\right\}_{n=1}^{\infty}$ converges to 0.

(3) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Suppose we know that $\{a_n\}_{n=1}^{\infty}$ converges to 1. Prove that there is a natural number $n \in \mathbb{N}$ such that $a_n > 0$.

> Take $\varepsilon = 1$. By definition of converges to 1, there is some N such that for all $n > N$, $|a_n - 1| < 1$, and in particular $a_n > 0$. So, take any natural number greater than n, and the conclusion follows.

(4) Prove or disprove: The sequence $\left\{\frac{n+1}{2n}\right\}_{n=1}^{\infty}$ converges to 0.

> Take $\varepsilon = 1/2$. We claim that there is no N such that for all $n > N$ we have $|a_n - 0| < 1/2$. Indeed, given N, take any n to be any natural number greater than N. Then $a_n = 1/2 + 1/2n > 1/2$, so $|a_n| > 1/2$. Thus, there is no N satisfying the desired property. This means that the sequence does not converge to 1/2.

(5) Prove or disprove: The sequence $\{a_n\}_{n=1}^{\infty}$ where

$$a_n = \begin{cases} 1 & \text{if } n = 10^m \text{ for some } m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

converges to 0.

> Take $\varepsilon = 1/2$. We claim that there is no N such that for all $n > N$ we have $|a_n - 0| < 1/2$. Indeed, let N be any real number. Let m be a natural number larger than N, and $n = 10^m$. Then $n = 10^m > m > N$, and $a_n = 1$, so $|a_n - 0| = 1 > 1/2$. This shows the claim, and hence that the sequence does not converge to 0

Definition 8.1. A sequence $\{a_n\}_{n=1}^{\infty}$ is *convergent* if there is a real number L such that $\{a_n\}_{n=1}^{\infty}$ converges to L. Otherwise, it is said to be *divergent*.

(6) In this problem, we will prove that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent.
- Proceed by contradiction and suppose it converges to L.
- Apply the definition of "converges to L" with $\varepsilon = \frac{1}{2}$, so we get some N.
- Take an odd integer n bigger than N: what does this say about L?
- Take an even integer n bigger than N: what does this say about L?
- Conclude the proof.

Proof. We proceed by contradiction: Suppose the sequence did converge to some number L. Our strategy will be to derive a contradiction by showing that such an L would have to satisfy mutually exclusive conditions.

By definition, since the sequence converges to L, we have that for every $\varepsilon > 0$ there is a number N such that $|(-1)^n - L| < \varepsilon$ for all natural numbers n such that $n > N$. In particular, this statement is true for the particular value $\varepsilon = \frac{1}{2}$. That is, there is a number N such that $|(-1)^n - L| < \frac{1}{2}$ for all natural numbers n such that $n > N$. Let n be any even natural number that is bigger than N. (Certainly one exists: we know there is an integer bigger than N by Theorem 5.3. Pick one. If it is even, take that to be n. If it is odd, increase it by one to get an even integer n.) Since $(-1)^n = 1$ for an even integer n, we get

$$|1 - L| < \frac{1}{2}$$

and thus $\frac{1}{2} < L < \frac{3}{2}$.

Likewise, let n be an odd natural number bigger than N. Since $(-1)^n = -1$ for an odd integer n, we get

$$|-1 - L| < \frac{1}{2}$$

and thus $-\frac{3}{2} < L < -\frac{1}{2}$. But it cannot be that both $L > \frac{1}{2}$ and $L < -\frac{1}{2}$.

We conclude that no such L exists; that is, this sequence is divergent. $\square$

9. September 20, 2022

Example 9.1. Let's prove the sequence $\{(-1)^n\}_{n=1}^{\infty}$ is divergent. This means that there is no L to which it converges.

Proof. We proceed by contradiction: Suppose the sequence did converge to some number L. Our strategy will be to derive a contradiction by showing that such an L would have to satisfy mutually exclusive conditions.

By definition, since the sequence converges to L, we have that for every $\varepsilon > 0$ there is a number N such that $|(-1)^n - L| < \varepsilon$ for all natural numbers n such that $n > N$. In particular, this statement is true for the particular value $\varepsilon = \frac{1}{2}$. That is, there is a number N such that $|(-1)^n - L| < \frac{1}{2}$ for all natural numbers n such that $n > N$. Let n be any even natural number that is bigger than N. (Certainly one exists: we know there is an integer bigger than N by Theorem 5.3. Pick one. If it is even, take that to be n. If it is odd, increase it by one to get an even integer n.) Since $(-1)^n = 1$ for an even integer n, we get

$$|1 - L| < \frac{1}{2}$$

and thus $\frac{1}{2} < L < \frac{3}{2}$.

Likewise, let n be an odd natural number bigger than N. Since $(-1)^n = -1$ for an odd integer n, we get

$$|-1 - L| < \frac{1}{2}$$

and thus $-\frac{3}{2} < L < -\frac{1}{2}$. But it cannot be that both $L > \frac{1}{2}$ and $L < -\frac{1}{2}$.

We conclude that no such L exists; that is, this sequence is divergent. □

Proposition 9.2. *If a sequence converges, then there is a unique number to which it converges.*

Proof. Recall that to show something satisfying certain properties is unique, one assumes there are two such things and argues that they must be equal. So, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence that converges to L and that also converges to M. We will prove $L = M$.

By way of contradiction, suppose $L \neq M$. Then set $\varepsilon = \frac{|L-M|}{3}$. Since we are assuming $L \neq M$, we have $\varepsilon > 0$. According to the definition of convergence, since the sequence converges to L, there is a real number N_1 such that for $n \in \mathbb{N}$ such that $n > N_1$ we have

$$|a_n - L| < \varepsilon.$$

Also according to the definition, since the sequence converges to M, there is a real number N_2 such that for $n \in \mathbb{N}$ and $n > N_2$ we have

$$|a_n - M| < \varepsilon.$$

Pick n to be any natural number larger than $\max\{N_1, N_2\}$ (which exists by Theorem 5.3). For such an n, both $|a_n - L| < \varepsilon$ and $|a_n - M| < \varepsilon$ hold. Using the triangle inequality and these two inequalities, we get

$$|L - M| \leq |L - a_n| + |M - a_n| < \varepsilon + \varepsilon.$$

But by the choice of ε, we have $\varepsilon + \varepsilon = \frac{2}{3}|L - M|$. That is, we have deduced that $|L - M| < \frac{2}{3}|L - M|$ which is impossible. We conclude that $L = M$. ☐

From now on, given a sequence $\{a_n\}_{n=1}^{\infty}$ and a real number L, will we use the short-hand notation

$$\lim_{n\to\infty} a_n = L$$

to mean that the given sequence converges to the given number. For example, we showed above that

$$\lim_{n\to\infty} \frac{2n-1}{5n+1} = \frac{2}{5}.$$

But, to be clear, the statement "$\lim_{n\to\infty} a_n = L$" signifies nothing more and nothing less than the statement "$\{a_n\}_{n=1}^{\infty}$ converges to L".

Here is some terminology we will need:

Definition 9.3. Suppose $\{a_n\}_{n=1}^{\infty}$ is any sequence.

(1) We say $\{a_n\}_{n=1}^{\infty}$ is *bounded above* if there exists at least one real number M such that $a_n \leq M$ for all $n \in \mathbb{N}$; we say $\{a_n\}_{n=1}^{\infty}$ is *bounded below* if there exists at least one real number m such that $a_n \geq m$ for all $n \in \mathbb{N}$; and we say $\{a_n\}_{n=1}^{\infty}$ is *bounded* if it is both bounded above and bounded below.

(2) We say $\{a_n\}_{n=1}^{\infty}$ is *increasing* if for all $n \in \mathbb{N}$, $a_n \leq a_{n+1}$; we say $\{a_n\}_{n=1}^{\infty}$ is *decreasing* if for all $n \in \mathbb{N}$, $a_n \geq a_{n+1}$; and we say $\{a_n\}_{n=1}^{\infty}$ is *monotone* if it is either decreasing or increasing.

(3) We say $\{a_n\}_{n=1}^{\infty}$ is *strictly increasing* if for all $n \in \mathbb{N}$, $a_n < a_{n+1}$. I leave the definition of *strictly decreasing* and *strictly monotone* to your imaginations.

Remark 9.4. Be sure to interpret "monotone" correctly. It means

$$(\forall n \in \mathbb{N}, a_n \leq a_{n+1}) \text{ or } (\forall n \in \mathbb{N}, a_n \geq a_{n+1});$$

it does *not* mean

$$\forall n \in \mathbb{N}, (a_n \leq a_{n+1}) \text{ or } (a_n \geq a_{n+1}).$$

Do you see the difference?

Proposition 9.5. *If a sequence* $\{a_n\}_{n=1}^{\infty}$ *converges then it is bounded.*

Proof. Suppose the sequence $\{a_n\}_{n=1}^{\infty}$ converges to the number L. Applying the definition of "converges to L" using the particular value $\varepsilon = 1$ gives the following fact: There is a real number N such that if $n \in \mathbb{N}$ and $n > N$, then $|a_n - L| < 1$. The latter inequality is equivalent to $L - 1 < a_n < L + 1$ for all $n > N$.

Let m be any natural number such that $m > N$, and consider the finite list of numbers

$$a_1, a_2, \ldots, a_{m-1}, L + 1.$$

Let b be the largest element of this list. I claim the sequence is bounded above by b. For any $n \in \mathbb{N}$, if $1 \leq n \leq m - 1$, then $a_n \leq b$ since in this case a_n is a member of the above list and b is the largest element of this list. If $n \geq m$ then since $m > N$, we have $n > N$ and hence $a_n < L + 1$ from above. We also have $L + 1 \leq b$ (since $L + 1$ is in the list) and thus $a_n < b$. This proves $a_n \leq b$ for all n as claimed.

Now take p to be the smallest number in the list

$$a_1, a_2, \ldots, a_{m-1}, L - 1.$$

A similar argument shows that $a_n \geq p$ for all $n \in \mathbb{N}$. □

10. September 22, 2022

(1) For each of the following sequences which of the following adjectives apply: bounded above, bounded below, bounded, (strictly) increasing, (strictly) decreasing, (strictly) monotone?
   (a) $\{\frac{1}{n}\}_{n=1}^{\infty}$
   (b) The Fibonacci sequence $\{f_n\}_{n=1}^{\infty}$ where $f_1 = f_2 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$.
   (c) $\{(-1)^n\}_{n=1}^{\infty}$
   (d) $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$.

(a) bounded, strictly decreasing, strictly monotone
(b) bounded below, increasing, monotone
(c) bounded
(d) bounded

(2) Prove or disprove the converse to Proposition 9.5.

The sequence $\{(-1)^n\}_{n=1}^{\infty}$ is bounded but divergent, so the converse is false.

Example 10.1. (1) A constant sequence $\{c\}_{n=1}^{\infty}$ converges to c.
(2) The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0.

Theorem 10.2 (Limits and algebra)**.** *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L, and $\{b_n\}_{n=1}^{\infty}$ be a sequence that converges to M.*

(1) If c is any real number, then $\{ca_n\}_{n=1}^{\infty}$ converges to cL.
(2) The sequence $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$.
(3) The sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges to LM.
(4) If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{1}{a_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{1}{L}$.
(5) If $M \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{L}{M}$.

(end of theorem, back to problems)

(3) Use Theorem 10.2 and Example 10.1 to show that the sequence $\{2 + 5/n - 7/n^2\}_{n=1}^{\infty}$ converges to 2.

The constant sequence $\{2\}_{n=1}^{\infty}$ converges to 2 by Ex 10.1 part 1. The sequence $\{5/n\}_{n=1}^{\infty} = \{5 \cdot 1/n\}_{n=1}^{\infty}$ converges to $5 \cdot 0 = 0$ by Ex 10.1 part 2 and Thm 10.2 part 1. The sequence $\{-7/n^2\}_{n=1}^{\infty} = \{-7 \cdot 1/n \cdot 1/n\}_{n=1}^{\infty}$ converges to $-7 \cdot 0 \cdot 0 = 0$ by Ex 10.1 part 2, Thm 10.2 part 1, and Thm 10.2 part 3. Thus, by Thm 10.2 part 2, the sequence $\{2+5/n-7/n^2\}_{n=1}^{\infty}$ converges to $2+0+0 = 2$ by Thm 10.2 part 2.

(4) Use Theorem 10.2 and Example 10.1 to show that the sequence $\frac{2n+3}{3n-4}$ converges to $\frac{2}{3}$.

(5) Use Theorem 10.2 to show that if $\{a_n\}_{n=1}^{\infty}$ converges to L, and $\{b_n\}_{n=1}^{\infty}$ converges to M, then $\{a_n - b_n\}_{n=1}^{\infty}$ converges to $L - M$.

> By Thm 10.2 part 1, $\{-b_n\}_{n=1}^{\infty}$ converges to $-M$. Then by Thm 10.2 part 2, $\{a_n - b_n\}_{n=1}^{\infty} = \{a_n + (-b_n)\}_{n=1}^{\infty}$ converges $L + (-M) = L - M$.

(6) Prove or disprove the following converse to part (2): If $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$ then $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{b_n\}_{n=1}^{\infty}$ converges to M.

> Take $\{a_n\}_{n=1}^{\infty} = \{(-1)^n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty} = \{(-1)^{n+1}\}_{n=1}^{\infty}$. Then $\{a_n + b_n\}_{n=1}^{\infty} = \{0\}_{n=1}^{\infty}$ converges to 0, but neither $\{a_n\}_{n=1}^{\infty}$ nor $\{b_n\}_{n=1}^{\infty}$ converges.

(7) Prove part (1) of Theorem 10.2 in the special case $c = 2$ by following the following steps:
- Assume that $\{a_n\}_{n=1}^{\infty}$ converges to L.
- We now want to show that $\{2a_n\}_{n=1}^{\infty}$ converges to something. We know what we have to write next!
- Now we do some scratchwork: we want an N such that for $n > N$ we have $|2a_n - 2L| < \varepsilon$. Factor this to get some inequality with a_n. How can we use our assumption to get an N that "works"?
- Complete the proof.

(8) Prove part (1) of Theorem 10.2.
(9) Prove part (2) of Theorem 10.2.
(10) Prove part (3) of Theorem 10.2.

11. September 27, 2022

Last time we looked at:

Theorem 10.2 (Limits and algebra)**.** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence that converges to L, and $\{b_n\}_{n=1}^{\infty}$ be a sequence that converges to M.

(1) If c is any real number, then $\{ca_n\}_{n=1}^{\infty}$ converges to cL.
(2) The sequence $\{a_n + b_n\}_{n=1}^{\infty}$ converges to $L + M$.
(3) The sequence $\{a_n b_n\}_{n=1}^{\infty}$ converges to LM.
(4) If $L \neq 0$ and $a_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\frac{1}{a_n}\right\}_{n=1}^{\infty}$ converges to $\frac{1}{L}$.

(5) If $M \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$, then $\left\{\dfrac{a_n}{b_n}\right\}_{n=1}^{\infty}$ converges to $\dfrac{L}{M}$.

The following is another useful technique:

Theorem 11.1 (The "squeeze" principle)**.** *Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, and $\{c_n\}_{n=1}^{\infty}$ are three sequences such that $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ both converge to L, and $a_n \leq b_n \leq c_n$ for all n. Then $\{b_n\}_{n=1}^{\infty}$ also converges to L.*

Proof. Assume $\{a_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ both converge to L and that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. We need to prove $\{b_n\}_{n=1}^{\infty}$ converges to L.

Pick $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converges to L there is a number N_1 such that if $n \in \mathbb{N}$ and $n > N_1$ then $|a_n - L| < \varepsilon$ and hence $L - \varepsilon < a_n < L + \varepsilon$. Likewise, since $\{c_n\}_{n=1}^{\infty}$ converges to L there is a number N_2 such that if $n \in \mathbb{N}$ and $n > N_2$ then $L - \varepsilon < c_n < L + \varepsilon$. Let

$$N = \max\{N_1, N_2\}.$$

If $n \in \mathbb{N}$ and $n > N$, then $n > N_1$ and hence $L - \varepsilon < a_n$, and $n > N_2$ and hence $c_n < L + \varepsilon$, and also $a_n \leq b_n \leq c_n$. Combining these facts gives that for $n \in \mathbb{N}$ such that $n > N$, we have

$$L - \varepsilon < b_n < L + \varepsilon$$

and hence $|b_n - L| < \varepsilon$. This proves $\{b_n\}_{n=1}^{\infty}$ converges to L. $\square$

Example 11.2. We can use the Squeeze Theorem to give a short proof that $\{5 + (-1)^n \frac{1}{n}\}_{n=1}^{\infty}$ converges to 5. Note that Theorem 10.2 alone cannot be used in this example. However, from Theorem 10.2, it follows that $\{5 - \frac{1}{n}\}_{n=1}^{\infty}$ and $\{5 + \frac{1}{n}\}_{n=1}^{\infty}$ both converge to 5. Then, since

$$5 - \frac{1}{n} \leq 5 + (-1)^n \frac{1}{n} \leq 5 + \frac{1}{n}$$

for all n, our sequence also converges to 5.

When I introduced the Completeness Axiom, I mentioned that, heuristically, it is what tells us that the real number line doesn't have any holes. The next result makes this a bit more precise:

Theorem 11.3. *Every increasing, bounded above sequence converges.*

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence that is both bounded above and increasing.

(Commentary: In order to prove it converges, we need to find a candidate number L that it converges to. Since the set of numbers occurring in this sequence is nonempty and bounded above, this number is provided to us by the Completeness Axiom.)

Let S be the set of those real numbers that occur in this sequence. (This is technically different that the sequence itself, since sequences are allowed to have repetitions but sets are not. Also, sequences have an ordering to them, but sets do not.) The set S is clearly nonempty, and it is bounded above since we assume the sequence is bounded above. Therefore, by the Completeness Axiom, S has a supremum L. We will prove the sequence converges to L.

Pick $\varepsilon > 0$. Then $L - \varepsilon < L$ and, since L is the supremum, $L - \varepsilon$ is not an upper bound of S. This means that there is an element of S that is strictly bigger than $L - \varepsilon$. Every element of S is a member of the sequence, and so we get that there is an $N \in \mathbb{N}$ such that $a_N > L - \varepsilon$.

(We will next show that this is the N that "works". Note that, in the general definition of convergence of a sequence, N can be any real number, but in this proof it turns out to be a natural number.)

Let n be any natural number such that $n > N$. Since the sequence is increasing, $a_N \leq a_n$ and hence

$$L - \varepsilon < a_N \leq a_n.$$

Also, $a_n \leq L$ since L is an upper bound for the sequence, and thus we have

$$L - \varepsilon < a_n \leq L.$$

It follows that $|a_n - L| < \varepsilon$. We have proven the sequence converges to L. $\square$

Theorem 11.4 (Monotone Converge Theorem)**.** *Every bounded monotone sequence converges.*

Proof. If $\{a_n\}_{n=1}^{\infty}$ is increasing, then this is the content of Theorem 11.3. If $\{a_n\}_{n=1}^{\infty}$ is decreasing and bounded, consider the sequence $\{-a_n\}_{n=1}^{\infty}$. If $a_n \leq M$ for all n, then $-a_n \geq -M$ for all n, so $\{-a_n\}_{n=1}^{\infty}$ is bounded below. Also, since $a_n \geq a_{n+1}$ for all n, we have $-a_n \leq -a_{n+1}$ for all n, so $\{-a_n\}_{n=1}^{\infty}$ is increasing. Thus, by Theorem 11.3, $\{-a_n\}_{n=1}^{\infty}$ converges, say to L. Then by Theorem 10.2(1), $\{a_n\}_{n=1}^{\infty} = \{-(-a_n)\}_{n=1}^{\infty}$ converges to $-L$. $\square$

Example 11.5. Consider the sequence $\{a_n\}_{n=1}^{\infty}$ given by the formula

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

We will use the Monotone Convergence Theorem to prove that this sequence converges.

First, we need to see that the sequence is increasing. Indeed, for every n we have that $a_{n+1} = a_n + \frac{1}{a_{n+1}^2} \geq a_n$.

Next, we need to show that it is bounded above. Observe that

$$\begin{aligned}
a_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\
&\le 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1)n} \\
&= 1 + (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + \cdots + (\frac{1}{n-1} - \frac{1}{n}) \\
&= 1 + 1 - \frac{1}{n},
\end{aligned}$$

so we have $a_n \le 2$ for all n. This means that $\{a_n\}_{n=1}^{\infty}$ is bounded above by 2.

Hence, by the Monotone Convergence Theorem, $\{a_n\}_{n=1}^{\infty}$ converges. Leonhard Euler was particularly interested in this sequence, and was able to prove that it converges to $\frac{\pi^2}{6}$. This requires some other ideas, so we won't do that here.

12. September 29, 2022

Which of the following implications about sequences hold in general? Either mention a relevant theorem or give a counterexample.

(a) monotone $\Longrightarrow$ convergent
(b) convergent $\Longrightarrow$ bounded
(c) bounded + decreasing $\Longrightarrow$ convergent
(d) increasing + convergent $\Longrightarrow$ bounded
(e) convergent $\Longrightarrow$ monotone
(f) bounded $\Longrightarrow$ convergent

(a) False: $\{n\}_{n=1}^{\infty}$
(b) True: (Every convergent sequence is bounded.)
(c) True: Monotone Convergence Theorem
(d) True: (Every convergent sequence is bounded.)
(e) False: $\{\frac{(-1)^n}{n}\}_{n=1}^{\infty}$
(f) False: $\{(-1)^n\}_{n=1}^{\infty}$

It is sometimes useful to distinguish between sequences like $\{(-1)^n\}_{n=1}^{\infty}$ that diverge because they "oscillate", and sequences like $\{n\}_{n=1}^{\infty}$ that diverge because they "head toward infinity".

(I) In intuitive language, a sequence converges to L if no matter how close we want or sequence to be to L, all values past some point are at least that close. Intuitively, a sequence *diverges to* $+\infty$ if no matter how large we want our sequence to be, all values past some point are at least that large. Write a precise definition for a sequence to diverge to $+\infty$.

(II) Write a precise definition for a sequence to diverge to $-\infty$.

(I) A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to* $+\infty$ if for every $M \in \mathbb{R}$, there is some $N \in \mathbb{R}$ such that for every natural number $n > N$, we have $a_n > M$.

(II) A sequence $\{a_n\}_{n=1}^{\infty}$ *diverges to* $-\infty$ if for every $m \in \mathbb{R}$, there is some $N \in \mathbb{R}$ such that for every natural number $n > N$, we have $a_n < m$.

(1) Carefully write the logical negation of "$\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$" in simplified form.

There exists $M \in \mathbb{R}$ such that for every $N \in \mathbb{R}$, there exists a natural number $n > N$ with $a_n < M$.

(2) Use the definition to prove that the sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to $+\infty$.

Let $M \in \mathbb{R}$. [Scratchwork: We need some N such that if $n > N$ then $\sqrt{n} > M$. This inequality is equivalent to $n > M^2$, so take $N = M^2$.] Take $N = M^2$. Let $n > N$ be a natural number. Then $\sqrt{n} > \sqrt{N} = \sqrt{M^2} = |M| \geq M$. This shows that $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges to $+\infty$.

(3) Prove that if a sequence $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$ then it is not bounded above.

We prove the contrapositive. Suppose that $\{a_n\}_{n=1}^{\infty}$ is bounded above, say by M. Suppose, to obtain a contradiction that $\{a_n\}_{n=1}^{\infty}$ diverges to $+\infty$. Then applying the definition with the number M, we have that there is some N such that for all $n > N$, $a_n > M$. But htere is no n for which $a_n > M$, so this is a contradiction, so $\{a_n\}_{n=1}^{\infty}$ does not diverge to $+\infty$.

(4) Use (3) to show that if a sequence diverges to $+\infty$ then it diverges.

Since every converges sequence is bounded, the conclusion follows.

(5) Prove or disprove: If a sequence is not bounded above, then it diverges to $+\infty$.

A counterexample is given by the sequence $\{(-1)^n n\}_{n=1}^{\infty}$. It is not bounded above, since for any M, we can take an even natural number n larger than M, and for this number, $(-1)^n n = n > M$. It does not diverge to $+\infty$: take $M = 0$; for any $N \in \mathbb{R}$, there is an odd natural number n larger than N, and for this n, we have $(-1)^n n = -n < 0 = M$.

(6) Prove or disprove: If a sequence diverges to $+\infty$ then it is increasing.

A counterexample is given by the sequence given by $a_1 = 3$, $a_n = n$ for $n > 1$. It is not increasing since $a_1 = 3 > 2 = a_2$. However, it diverges to $+\infty$ since, given M, we can take $N = M$, and for any $n > N$, we have $a_n = n > N = M$.

(7) Prove or disprove: If a sequence is increasing and not bounded above, it diverges to ∞.

To prove it, let $\{a_n\}_{n=1}^{\infty}$ be increasing and not bounded above. Take $M \in \mathbb{R}$. Since it is not bounded above, there is some $N \in \mathbb{N}$ such that $a_N > M$. Then, for this N, for any $n > N$ we have $a_n \geq a_N$ since the sequence is increasing, so $a_n > M$. This shows the sequence diverges to $+\infty$.

13. October 4, 2022

We will now embark on a bit of detour. I've postponed talking about proofs by induction, but we will need to use that technique on occasion. So let's talk about that idea now.

The technique of proof by induction is used to prove that an infinite sequence of statements indexed by $\mathbb{N}$

$$P_1, P_2, P_3, \ldots$$

are all true. For example the equation

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

holds for all $n \in \mathbb{N}$. We get one statement for each natural number:

$$\begin{aligned}
P_1 : \qquad\qquad 1 &= \frac{1 \cdot 2}{2} \\
P_2 : \qquad\quad 1 + 2 &= \frac{2 \cdot 3}{2} \\
P_3 : \qquad 1 + 2 + 3 &= \frac{3 \cdot 4}{2} \\
\vdots \qquad\qquad \vdots &
\end{aligned}$$

Such a fact (for all n) is well-suited to be proven by induction.

Here is the general principle:

Theorem 13.1 (Principle of Mathematical Induction)**.** *Suppose we are given, for each $n \in \mathbb{N}$, a statment P_n. Assume that P_1 is true and that for each $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true. Then P_n is true for all $n \in \mathbb{N}$.*

"The domino analogy": Think of the statements $P_1, P_2, \ldots$ as dominoes lined up in a row. The fact that $P_k \implies P_{k+1}$ is interpreted as meaning that the dominoes are arranged well enough so that if one falls, then so does the next one in the line. The fact that P_1 is true is interpreted as meaning the first one has been knocked over. Given these assumptions, for every n, the n-th domino will (eventually) fall down.

The Principle of Mathematical Induction (PMI) is indeed a theorem, which we will now prove:

Proof. Assume that P_1 is true and that for each $k \in \mathbb{N}$, if P_k is true, then P_{k+1} is true. Consider the subset

$$S = \{n \in \mathbb{N} \mid \ P_n \text{ is false}\}$$

of $\mathbb{N}$. Our goal is to show S is the empty set.

By way of contradiction, suppose S is not empty. Then by the Well-Ordering Principle, S has a smallest element, call it ℓ. (In other words, P_ℓ is the first statement in the list $P_1, P_2, \ldots,$ that is false.) Since P_1 is true, we must have $\ell > 1$. But then $\ell - 1 < \ell$ and so $\ell - 1$ is not in S. Since $\ell > 1$, we have $\ell - 1 \in \mathbb{N}$ and thus we can say that $P_{\ell-1}$ must be true. Since $P_k \Rightarrow P_{k+1}$ for any k, letting $k = \ell - 1$, we see that, since $P_{\ell-1}$ is true, P_ℓ must also by true. This contradicts the fact that $\ell \in S$. We conclude that S must be the empty set. $\square$

The above proof shows that the Principle of Mathematical Induction is a consequence of the Well-Ordering Principle. The converse is also true.

Example 13.2. Let's prove that the formula

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

for every natural number n. Here, P_k is

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

For P_1 we have $1 = \frac{1 \cdot 2}{2}$ is true. Now we show P_k implies P_{k+1}. Let k be a natural number and assume that

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

Then

$$\begin{aligned} 1 + 2 + 3 + \cdots + k + (k+1) &= (1 + 2 + 3 + \cdots + k) + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}, \end{aligned}$$

which is P_{k+1}. Thus we have proven the equality for all natural numbers n by induction. $\square$

Example 13.3. Let us show that for every real number $x \geq -1$, and every natural number $n \in \mathbb{N}$, the inequality $(1+x)^n \geq 1 + nx$.

Fix a real number $x \geq -1$. We show that $(1+x)^n \geq 1 + nx$ for all natural numbers n by induction. For $n = 1$, we have

$$(1+x)^1 = 1 + x = 1 + 1 \cdot x,$$

so the statement is true for $n = 1$. Let k be a natural number and assume that

$$(1+x)^k \geq 1 + kx.$$

Then,

$$(1+x)^{k+1} = (1+x) \cdot (1+x)^k \geq (1+x)(1+kx) = 1 + (k+1)x + kx^2 \geq 1 + (k+1)x,$$

where we used that $1 + x \geq 0$ in the first $\geq$ (since $x \geq -1$) and that $x^2 \geq 0$ in the second $\geq$. Thus, by induction, the inequality is true for all $n \in \mathbb{N}$. $\square$

Induction is also closely related to the notion of a sequence defined by recursion. Recall that we define a sequence $\{a_n\}_{n=1}^\infty$ recursively by specifying a value a_1 for the first value, and a formula for a_n in terms of a_{n-1} (or multiple earlier values in the sequence). The principle of induction justifies that such a rule gives a well-defined value for every n: if we take P_k to be the statement that the formulas define value for all of the first k terms $a_1, \dots, a_k$, then P_1 is true and $P_k \Rightarrow P_{k+1}$, so P_n is true for every $n \in \mathbb{N}$.

Proposition 13.4. *For any real number r, there exists a strictly increasing sequence of rational numbers that converges to r.*

Proof. We will construct a sequence of rational numbers $\{q_n\}_{n=1}^\infty$ such that $r - \frac{1}{n} < q_n < r$ for every n that is strictly increasing, and then show that this sequence converges to r. By Density of Rational Numbers, there exists a rational number q_1 such that $r - 1 < q < r$. Given $q_1, \dots, q_n$, we define q_{n+1} recursively to be a rational number such that $\max\{r - \frac{1}{n+1}, q_n\} < q_{n+1} < r$ again using Density of Rational Numbers. To see that this rule makes sense, we observe that if we have constructed $q_1, \dots, q_n$ by this rule, then $q_n < r$, so $\max\{r - \frac{1}{n+1}, q_n\} < r$, and hence Density of Rational number applies, so we can construct q_{n+1} (and hence we can construct q_n for any n by this rule). Since

$$q_n \leq \max\{r - \frac{1}{n+1}, q_n\} < q_{n+1}$$

for every n, the sequence we obtain is strictly increasing. Since

$$r - \frac{1}{n} < q_n < r$$

for every n and $\{r - \frac{1}{n}\}_{n=1}^\infty$ converges to r, by the Squeeze Theorem, the sequence $\{q_n\}_{n=1}^\infty$ converges to r. □

14. October 6, 2022

Decimal expansions. In this worksheet, we are going to define decimal expansions and prove the basic properties about them. To simplify things, we are going to only deal with numbers between 0 and 1 (since we get all the the rest by adding integers and taking negatives). Along the way we will use induction and convergence of sequences in an important way. Before we define infinite decimal expansions, let's review finite decimal expansions.

(1) If $d \in \{0, 1, \dots, 9\}$ (i.e., d is an integer between 0 and 9), what does the decimal number $0.d$ mean? Express it as a rational number.

$$\boxed{\frac{d}{10}}$$

(2) If $d_1, d_2, \ldots, d_n \in \{0, 1, \ldots, 9\}$ (i.e., $d_1, \ldots, d_n$ are a bunch of integers between 0 and 9, which may or may not have repeats), convince yourself that the decimal number $0.d_1 d_2 \cdots d_n$ in the way that we commonly use it is shorthand for

$$0.d_1 d_2 \cdots d_n = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}.$$

Let's say that a sequence of the form $\{d_n\}_{n=1}^{\infty}$ is a *digit sequence* if $d_n \in \{0, 1, \ldots, 9\}$ for all n. (That is a digit sequence is just a sequence of integers between 0 and 9.) Given a digit sequence $\{d_n\}_{n=1}^{\infty}$, define another sequence $\{D_n\}_{n=1}^{\infty}$ by the rule

$$\begin{aligned} D_1 &= \frac{d_1}{10^1} \\ D_2 &= \frac{d_1}{10^1} + \frac{d_2}{10^2} \\ &\vdots \\ D_n &= \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n} \\ &\vdots \end{aligned}$$

For example, for the digit sequence $2, 2, 2, \ldots$, the corresponding $\{D_n\}_{n=1}^{\infty}$ sequence is

$$\frac{2}{10}\ , \ \frac{2}{10} + \frac{2}{100}\ , \ \frac{2}{10} + \frac{2}{100} + \frac{2}{1000}\ , \ \ldots$$

We say that a digit sequence $\{d_n\}_{n=1}^{\infty}$ *represents a real number* r if the sequence $\{D_n\}_{n=1}^{\infty}$ converges to r, and in this case we write

$$0.d_1 d_2 d_3 d_4 \cdots = r.$$

In order to prepare for proving things about decimal expansions, we need a fact about geometric series.

(1) Let x and a be real numbers.
   (a) Prove that for every $n \in \mathbb{N}$,

$$(1 - x)(1 + x + x^2 + x^3 + \cdots + x^n) = 1 - x^{n+1}.$$

We proceed by induction on n. First we check for $n = 1$:

$$(1-x)(1+x) = 1 + x - x - x^2 = 1 - x^{1+1},$$

so the statement is true for $n = 1$. Suppose the equality holds for k:

$$(1-x)(1+x+x^2+x^3+\cdots+x^k) = 1 - x^{k+1}.$$

Then

$$\begin{aligned}
&(1-x)(1+x+x^2+x^3+\cdots+x^{k+1}) \\
&\quad = (1-x)((1+x+x^2+x^3+\cdots+x^k)+(x^{k+1})) \\
&\quad = (1-x)(1+x+x^2+x^3+\cdots+x^k)+(1-x)(x^{k+1}) \\
&\quad = 1 - x^{k+1} + x^{k+1} - x^{k+2} = 1 - x^{k+2},
\end{aligned}$$

and it holds for $k+1$. Thus, the statement is true for all n by induction.

(b) If $x \neq 1$, use (a) to show that for every $n \in \mathbb{N}$,

$$a + ax + ax^2 + \cdots + ax^n = a\frac{1-x^{n+1}}{1-x}.$$

We have

$$a+ax+ax^2+\cdots+ax^n = a(1+x+x^2+x^3+\cdots+x^n) = a\frac{1-x^{n+1}}{1-x}.$$

(2) Use the definition (and perhaps the previous problem), but not our previous expectations about decimal expansions, to answer the following.
   (a) What number does the digit sequence $2, 3, 0, 0, 0, 0, 0, \ldots$ represent?
   (b) What number does the digit sequence $5, 0, 0, 0, 0, 0, 0, \ldots$ represent?
   (c) What number does the digit sequence $9, 9, 9, 9, 9, 9, 9, \ldots$ represent?
   (d) What number does the digit sequence $4, 9, 9, 9, 9, 9, 9, \ldots$ represent?

(a) $\frac{23}{100}$
(b) $\frac{5}{10} = \frac{1}{2}$

(c) $D_n = \frac{9}{10}\frac{1-(1/10)^{n+1}}{1-(1/10)} = 1-(\frac{1}{10})^{n+1}$ converges to 1, so this represents 1.

(d) $\frac{1}{2}$.

(3) Let $\{d_n\}_{n=1}^{\infty}$ be any digit sequence. Prove that this sequence represents some real number: i.e., that the corresponding sequence $\{D_n\}_{n=1}^{\infty}$ is convergent.

[Thus, every decimal expansion $0.d_1d_2d_3\cdots$ always gives us a real number.]

Note that the sequence D_n is increasing, and hence monotone. Since $d_n \le 9$, $D_n \le \frac{9}{10}\frac{1-(1/10)^{n+1}}{1-(1/10)} = 1-(\frac{1}{10})^{n+1} \le 1$, so it is bounded above. Thus D_n is always convergent.

(4) In this problem, we will show that every real number $r \in [0,1]$ is represented by some digit sequence.

(a) Show that we can recursively define a digit sequence $\{d_n\}_{n=1}^{\infty}$ such that for every $n \in \mathbb{N}$, in the corresponding sequence $\{D_n\}_{n=1}^{\infty}$, we have $0 \le 10^n(r-D_n) \le 1$.

(b) Given a sequence as in part (a), show that $\{D_n\}_{n=1}^{\infty}$ converges to r.

[Thus, every number can be written as a decimal expansion $0.d_1d_2d_3\cdots$.]

Since $0 \le 10r \le 10$, we can take an integer between 0 and 9 to be d_1 with $d_1 \le r \le d_1+1$, so $0 \le r-d_1 \le 1$. If we have chosen $d_1,\dots,d_n$ with $0 \le 10^n(r-D_n) \le 1$, then $0 \le 10^{n+1}(r-D_n) \le 10$ so we can choose d_{n+1} with $d_{n+1} \le 10^{n+1}(r-D_n) \le d_{n+1}+1$, and hence $0 \le 10^{n+1}(r-D_n)-d_{n+1} \le 1$. We then just need to note that $10^{n+1}(r-D_n)-d_{n+1} = 10^{n+1}D_{n+1}$.

(5) Now we analyze uniqueness of decimal expansions. We will find it useful to use the following corollary of the proof of the Monotone Convergence Theorem: If $\{a_n\}_{n=1}^{\infty}$ is a bounded increasing sequence, $\{a_n\}_{n=1}^{\infty}$ converges to $\sup(\{a_n \mid n \in \mathbb{N}\})$.

(a) Let $\{d_n\}_{n=1}^{\infty}$ be any digit sequence, $\{D_n\}_{n=1}^{\infty}$ be the corresponding sequence, and r the number that it represents. Let n be a natural number.

(i) Show that $D_n \leq r$ and that $D_n = r$ if and only if $d_i = 0$ for all $i > n$.

(ii) Show that $r \leq D_n + \frac{1}{10^n}$ and that $r = D_n + \frac{1}{10^n}$ if and only if $d_i = 9$ for all $i > n$.

(b) Let $\{d_n\}_{n=1}^{\infty}$ and $\{e_n\}_{n=1}^{\infty}$ be two digit sequences with $d_k \neq e_k$ for some $k \in \mathbb{N}$. Suppose that both digit sequences represent the same number r. Show that $r = \frac{m}{10^k}$ for some natural number m.

(c) Deduce that if $r \in [0, 1]$ and r cannot be written as a rational number with denominator a power of ten, then there is a unique digit sequence that represents r.
[Thus, if r cannot be written as a rational number with denominator a power of ten, then r has a unique decimal expansion.]

(d) Show that if $r \in (0, 1)$ and r can be written as a rational number with denominator a power of ten, then there are exactly two digit sequences that represent r: one with $d_i = 0$ for all i greater than some k, and one with $d_i = 9$ for all i greater than some k.
[Thus, if r has at most two decimal expansions, and always has exactly one nonterminating decimal expansion.]

15. October 20, 2022

We next discuss the important concept of a "subsequence".

Informally speaking, a subsequence of a given sequence is a sequence one forms by skipping some of the terms of the original sequence. In other words, it is a sequence formed by taking just some of the terms of the original sequence, but still infinitely many of them, without repetition.

We'll cover the formal definition soon, but let's give a few examples first, based on this informal definition.

Example 15.1. Consider the sequence

$$a_n = \begin{cases} 7 & \text{if } n \text{ is divisible by 3 and} \\ \frac{1}{n} & \text{if } n \text{ is not divisible by 3.} \end{cases}$$

If we pick off every third term starting with the term a_3 we get the subsequence

$$a_3, a_6, a_9, \ldots$$

which is the constant sequence

$$7, 7, 7, \ldots.$$

If we pick off the other terms we form the subsequence

$$a_1, a_2, a_4, a_5, a_7, a_8, a_{10}, \ldots$$

which gives the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{10}, \ldots.$$

Note that it is a little tricky to find an explicit formula for this sequence.

Example 15.2. For another, simpler, example, consider the sequence $\{(-1)^n \frac{1}{n}\}_{n=1}^{\infty}$. Taking just the odd-indexed terms gives the sequence

$$-1, -\frac{1}{3}, -\frac{1}{5}, -\frac{1}{7}, -\frac{1}{9}, \ldots$$

and taking the even-indexed terms gives the sequence

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \ldots$$

This time we can easily give a formula for each of these sequences: the first is

$$\{-\frac{1}{2n-1}\}_{n=1}^{\infty}$$

and the second is

$$\{\frac{1}{2n}\}_{n=1}^{\infty}.$$

Here is the formal definition:

Definition 15.3. A *subsequence* of a given sequence $\{a_n\}_{n=1}^{\infty}$ is any sequence of the form

$$\{a_{n_k}\}_{k=1}^{\infty}$$

where

$$n_1, n_2, n_3, \ldots$$

is any strictly increasing sequence of natural numbers — that is $n_k \in \mathbb{N}$ and $n_{k+1} > n_k$ for all $k \in \mathbb{N}$, so that

$$n_1 < n_2 < n_3 < \cdots .$$

Note that k is the index of the subsequence; i.e., the first term in the subsequence is when $k = 1$, the second is when $k = 2$ and so on. The integer sequence $\{n_k\}_{k=1}^{\infty}$ is the sequence of indices of the original sequence we choose to make the subsequence.

Example 15.4. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.

Setting $n_k = 2k - 1$ for all $k \in \mathbb{N}$ gives the subsequence of just the odd-indexed terms of the original sequence.

Setting $n_k = 2k$ for all $k \in \mathbb{N}$ gives the subsequence of just the even-indexed terms of the original sequence.

Setting $n_k = 3k - 2$ for all $k \in \mathbb{N}$ gives the subsequence of consising of every third term of the original sequence, starting with the first.

Setting $n_k = 100 + k$ gives the subsequence that is that "tail end" of the original, obtained by skipping the first 100 terms:

$$a_{101}, a_{102}, a_{103}, a_{104}, \ldots.$$

Of course, there is nothing special about 100 in this example.

The following result is important:

Theorem 15.5. *If a sequence $\{a_n\}_{n=1}^{\infty}$ converges to L, then every subsequence of this sequence also converges to L.*

We prepare with a lemma.

Lemma 15.6. *Let $b_1, b_2, \ldots$ be any strictly increasing sequence of natural numbers; that is, assume $b_k \in \mathbb{N}$ for all $k \in \mathbb{N}$ and that $b_k < b_{k+1}$ for all $k \in \mathbb{N}$. Then $b_k \geq k$ for all k.*

Proof. Suppose $b_1, b_2, \ldots$ is a strictly increasing sequence of natural numbers. We prove $b_n \geq n$ for all n by induction on n. That is, for each $n \in \mathbb{N}$, let P_n be the statement that $b_n \geq n$.

P_1 is true since $b_1 \in \mathbb{N}$ and so $b_1 \geq 1$. Given $k \in \mathbb{N}$, assume P_k is true; that is, assume $b_k \geq k$. Since $b_{k+1} > b_k$ and both are natural numbers, we have $b_{k+1} \geq b_k + 1 \geq k + 1$; that is, P_{k+1} is true too. By induction, P_n is true for all $n \in \mathbb{N}$. □

Proof of Theorem 15.5. Let the sequence $\{a_n\}_{n=1}^{\infty}$ converge to L, and take a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ for some strictly increasing sequence $n_1 < n_2 < n_3 < \cdots$ of natural numbers.

Let $\varepsilon > 0$. Since $\{a_n\}_{n=1}^{\infty}$ converges to L, there is some $N \in \mathbb{R}$ such that for all natural numbers $n > N$ we have $|a_n - L| < \varepsilon$. We claim that the same N works to verify the definition of $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L for this ε. Indeed, if $k > N$, then $n_k > N$, so $|a_{n_k} - L| < \varepsilon$. Thus, $\{a_{n_k}\}_{k=1}^{\infty}$ converges to L. □

(1) **True or false; justify.**
  (a) The sequence $\left\{\dfrac{1}{2n}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$.
  (b) The sequence $\left\{\dfrac{1}{3n+7}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\dfrac{1}{n}\right\}_{n=1}^{\infty}$.

(c) The constant sequence $\left\{\frac{1}{2}\right\}_{n=1}^{\infty}$ is a subsequence of the sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$.

(d) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are both subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

(e) The constant sequences $\{-1\}_{n=1}^{\infty}$ and $\{1\}_{n=1}^{\infty}$ are the only two subsequences of the sequence $\{(-1)^n\}_{n=1}^{\infty}$.

(a) True: take $n_k = 2k$.
(b) True: Take $n_k = 3k + 7$.
(c) False: The term $1/2$ occurs only for $n = 2$, so we can't choose an increasing sequence of indices that yield this value.
(d) True: take take $n_k = 2k + 1$ and take $n_k = 2k$, respectively.
(e) False: The sequence itself is a subsequence ($n_k = k$).

(2) Explain how the following Corollary follows from Theorem 15.5.
Corollary 15.7: Let $\{a_n\}_{n=1}^{\infty}$ be any sequence.
(a) If there is a subsequence of this sequence that diverges, then the sequence itself diverges.
(b) If there are two subsequences of this sequence that converge to different values, then the sequence itself diverges.

These are special cases of the contrapositive.

(3) Use Corollary 15.7 to give a quick proof that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges.

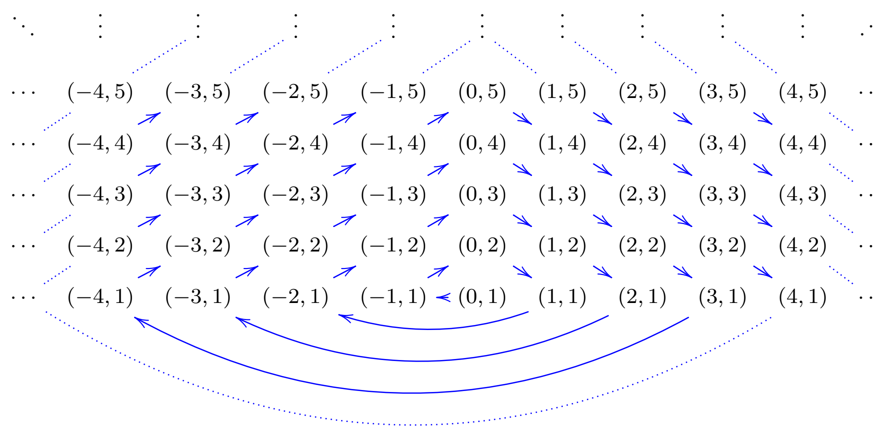
It has subsequences that converge to different values.

(4) **Prove or disprove:**
(a) Every subsequence of a bounded sequence is bounded.
(b) Every subsequence of a divergent sequence is divergent.
(c) Every subsequence of a sequence that diverges to $-\infty$ also diverges to $-\infty$.

(a) True: if $m < a_n < M$ for all n and $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence of natural numbers, then $m < a_{n_k} < M$ for all k.
(b) False: The divergent sequence $\{(-1)^n\}_{n=1}^{\infty}$ has a convergent subsequence $\{1\}_{n=1}^{\infty}$.
(c) True: Let $n_1 < n_2 < n_3 < \cdots$ be a strictly increasing sequence of natural numbers, and $\{a_{n_k}\}_{k=1}^{\infty}$ is a subseqence. Let $m \in \mathbb{R}$. There is some N such that $a_n < m$ for all $n > N$. We claim that this N works (for this m) to show that $\{a_{n_k}\}_{k=1}^{\infty}$ diverges to $-\infty$. Indeed, if $k > N$, then $n_k \geq k > N$, so $a_{n_k} < m$. Thus $\{a_{n_k}\}_{k=1}^{\infty}$ diverges to $-\infty$.

16. October 25, 2022

Consider the points in the plane whose x-coordinates are integers and y-coordinates are natural numbers. Starting at $(0,1)$, zigzag like so:



This gives the list of points

$(0,1),(-1,1),(0,2),(1,1),(-2,1),(-1,2),(0,3),(1,2),(2,1),(-3,1),\ldots$

Now convert these to a list of rational numbers by changing (m,n) to $\frac{m}{n}$ to get the sequence

$$\frac{0}{1},\frac{-1}{1},\frac{0}{2},\frac{1}{1},\frac{-2}{1},\frac{-1}{2},\frac{0}{3},\frac{1}{2},\frac{2}{1},\frac{-3}{1},\ldots$$

of rational numbers. Call this sequence $\{w_n\}_{n=1}^{\infty}$. (Even though you didn't want to know, we can give w_n by a formula as

$$w_n=\begin{cases}\dfrac{n-t^2+t-1}{n-t^2+2t-1} & \text{if } n\le t^2-t\\[2ex] \dfrac{-n+t^2-t+1}{-n+t^2+1} & \text{if } n> t^2-t\end{cases},$$

where $t=\min\{m\in\mathbb{N}\mid m^2\ge n\}$.)

Proposition 16.1. *There is a sequence $\{w_n\}_{n=1}^{\infty}$ of rational numbers such that*

1. *every rational number occurs in $\{w_n\}_{n=1}^{\infty}$ infinitely many times;*
2. *every sequence of rational numbers is a subsequence of $\{w_n\}_{n=1}^{\infty}$; and*
3. *every real number occurs as the limit of some subsequence of $\{w_n\}_{n=1}^{\infty}$.*

Proof. (1) The idea is that every point (m, n) with $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ gets passed through by the zigzag at least once. Then, given any rational number $q = a/b$, we can choose $b > 0$ by replacing a and b by their negatives if necessary. Then $q = \frac{a}{b} = \frac{2a}{2b} = \frac{3a}{3b} = \cdots$, so it comes from infinitely many pairs, and hence occurs infinitely many times.

(2) Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of rational numbers. We will realize it as a subsequence of $\{w_n\}_{n=1}^{\infty}$ by constructing an increasing sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$ such that $w_{n_k} = q_k$. Since $q_1 \in \mathbb{Q}$, there is some $n_1 \in \mathbb{N}$ such that $w_{n_1} = q_1$ by part (a). Suppose that we have defined $n_1 < \cdots < n_t$ such that $w_{n_k} = q_k$ for $k = 1, \ldots, t$. We claim that there is some n_{k+1} such that $n_{k+1} > n_k$ and $w_{n_{k+1}} = q_{k+1}$. Indeed, there are infinitely many $m \in \mathbb{N}$ such that $w_m = q_{k+1}$ by part (a), so at least one of these values of m is greater than n_k (since there are only fintiely many natural numbers less than or equal to n_k). Taking n_{k+1} to be m, we can continue the sequence, and we thus obtain such a sequence by recursion.

(3) Given $r \in \mathbb{R}$, we know that there exists a sequence of rational numbers that converges to r (moreover, there exists a strictly increasing one). This sequence can be obtained as a subsequence of $\{w_n\}_{n=1}^{\infty}$ by part (b), so we are done. □

On the other hand, there is no sequence that actually contains every real number. To prove this, we will use decimal expansions, as discussed earlier.

Recall that if $d_1, d_2, d_3, \ldots$ is a sequence of "digits", where $d_i \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ for every i, then the sequence $\{D_n\}_{n=1}^{\infty}$, where

$$D_n = \frac{d_1}{10^1} + \frac{d_2}{10^2} + \cdots + \frac{d_n}{10^n}$$

converges, and we say that $.d_1 d_2 d_3 \cdots$ is a *decimal expansion* for the real number $r = \lim_{n\to\infty} D_n$.

Theorem 16.2 (Cantor's Theorem)**.** *There is no sequence that contains every real number.*

Proof. By way of contradiction, suppose $\{a_n\}_{n=1}^{\infty}$ is a sequence in which every real number appears at least once. Write each member of this sequence in its decimal form, so that

$$\begin{aligned} a_1 &= (\text{integer part}).d_{1,1}d_{1,2}d_{1,3}\cdots \\ a_2 &= (\text{integer part}).d_{2,1}d_{2,2}d_{2,3}\cdots \\ a_3 &= (\text{integer part}).d_{3,1}d_{3,2}d_{3,3}\cdots \\ &\vdots \end{aligned}$$

where each $d_{i,j} \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is a digit. Now form a real number x as $0.e_1e_2e_3\cdots$ where the e_i's are digits chosen as follows: Let

$$e_i = \begin{cases} 7 & \text{if } d_{i,i} \leq 5 \\ 3 & \text{if } d_{i,i} > 5. \end{cases}$$

In particular, $e_i \neq d_{i,i}$ for every i. This means that the digit sequence $e_1, e_2, e_3, \ldots$ is not equal to any of the other digit sequences $d_{i,1}, d_{i,2}, d_{i,3}, \ldots$ for any i, because the i-th values are different. Moreover, the number x has a unique decimal expansion (since the only time two decimal expansions give the same number is one is eventually all 0's and the other is eventually all 9's), so $a_i \neq x$ for all $i \in \mathbb{N}$.

Thus x is not a member of this sequence, contrary to what we assumed. □

Our next big theorem has a very short statement, but is surprisingly tricky to prove.

Theorem 16.3 (Bolzano-Weierstrass Theorem)**.** *Every sequence has a monotone subsequence.*

The proof of this theorem requires a preliminary lemma.

Lemma 16.4. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.*

(1) If the set of values of the sequence $\{a_n \mid n \in \mathbb{N}\}$ does not have a maximum value, then $\{a_n\}_{n=1}^{\infty}$ has a subsequence that is increasing.

(2) If the set of values of the sequence $\{a_n \mid n \in \mathbb{N}\}$ does not have a minimum value, then $\{a_n\}_{n=1}^{\infty}$ has a subsequence that is decreasing.

Proof. (1) Assume that the set of values $\{a_n \mid n \in \mathbb{N}\}$ does not have a maximum value.

We define a subsequence recursively. We will recursively choose natural numbers $n_1, n_2, n_3, \ldots$ so that $n_k < n_{k+1}$ for all k and $a_{n_k} \leq a_{n_{k+1}}$.

We start by setting $n_1 = 1$.

If we have chosen n_k, then let $b = \max\{a_1, \ldots, a_{n_k}\}$.

We claim that there is some $m > n_k$ such that $a_m > b$. To obtain a contradiction, suppose otherwise. Then for any $n \in \mathbb{N}$, either $n > n_k$ and $a_n \leq b$ by assumption, or $n \leq n_k$ and $a_n \leq b$, since a_n is on the list of things of which b was the maximum. Then b is the maximum of $\{a_n \mid n \in \mathbb{N}\}$, which yields a contradiction. Thus, there is some $m > n_k$ such that $a_m > b$, and we can choose $m = n_{k+1}$. Thus, we can define such a sequence recursively.

(2) Similar to (1), or apply (1) to $\{-a_n\}_{n=1}^{\infty}$. □

Proof of Bolzano-Weierstrass Theorem 16.3. Let $\{a_n\}_{n=1}^{\infty}$ be any sequence. Recall that our goal is to prove it either has an increasing subsequence or it has a decreasing subsequence. This is equivalent to showing that if it has no increasing subsequences, then it does have at least one decreasing subsequence. So, let us assume it has no increasing subsequences.

We will prove it has at least one decreasing subsequence by constructing the indices $n_1 < n_2 < \cdots$ of such a subsequence recursively. By the contrapositive of part (1) Lemma 16.4 , since $\{a_n\}_{n=1}^{\infty}$ does not contain any increasing subsequences, we know that $\{a_n \mid n \in \mathbb{N}\}$ has a maximum value. That is, there exists a natural number n_1 such that $a_{n_1} \geq a_m$ for all $m \geq 1$.

For any k, given n_k, the subsequence $a_{n_k+1}, a_{n_k+2}, a_{n_k+3}, \ldots$ also has no increasing subsequence, since a subsequence of such a sequence is a subsequence of the original sequence too. Thus, it must have a maximum value again by part (1) Lemma 16.4; choose n_{k+1} such that $a_{n_{k+1}} = \max\{a_{n_k+1}, a_{n_k+2}, a_{n_k+3}, \ldots\}$. By construction, we have $n_{k+1} > n_k$. Thus, this gives a recursive definition for n_k.

For any k, note that a_{n_k} is the maximum of a set that contains $a_{n_{k+1}}$ (since it is later in the sequence). It follows that $a_{n_k} \geq a_{n_{k+1}}$. That is, we have constructed a decreasing subsequence of the original sequence. □

Corollary 16.5 (Main Corollary of Bolzano-Weierstrass Theorem)**.** *Every bounded sequence has a convergent subsequence.*

Proof. Suppose $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence. By the Bolzano-Weierstrass Theorem 16.3 it admits a monotone subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, and it too is bounded (since any subsequence of a bounded sequence is also bounded.) The result follows since every monotone bounded sequence converges by the Monotone Convergence Theorem 11.4. □

17. October 27, 2022

You can use any basic trig facts below to answer the following questions.

(1) **Explain but don't prove:** Is $\{\cos(\pi n)\}_{n=1}^{\infty}$ a subsequence of $\{\cos(n)\}_{n=1}^{\infty}$?

(2) **Prove or disprove:** The sequence $\{\cos(n)\}_{n=1}^{\infty}$ has a convergent subsequence.

(3) **Prove or disprove:** The sequence $\{\cos(n)\}_{n=1}^{\infty}$ has a constant subsequence.

(4) **Prove or disprove:** The sequence $\{\cos(n)\}_{n=1}^{\infty}$ has a subsequence that converges to some $x > 1$.

(a) No; to get a subsequence we would need have natural numbers inside the cosine, not multiples of π.
(b) True: $\cos(n)$ is bounded, so there is a convergent subsequence by Main Corollary of Bolzano-Weierstrass.
(c) False: in fact, $\cos(n)$ never takes the same value twice. If it did, we would have $\cos(n) = \cos(m)$ for natural numbers $m \neq n$, so $m - n = 2\pi k$ or $m + n = 2\pi k$, for some integer k, which would make $\pi = \frac{m-n}{2k}$ or $\pi = \frac{m+n}{2k}$, contradicting that π is irrational.
(d) False: if there is a subsequence converging to $x > 1$, let $\varepsilon = x - 1 > 0$. Then for some K, for all $k > K$, $|\cos(n_k) - 1| < \varepsilon$, which implies $\cos(n_k) > 1$, which is a contradiction.

Given any two sets S and T, a *function* from S to T, written $f : S \to T$, is a "rule" that assigns to each element $s \in S$ a unique element $t \in T$. The set S is called the *domain* of f. We will generally consider functions from some set of real numbers to $\mathbb{R}$. We often specify functions by formulas; when we do this the take the domain to be the set of all real numbers for which the formula evaluates to a unique real number. In

Here's a real definition: a *function* from S to T is a subset $G \subset S \times T$ of ordered pairs of elements of S and T with the property that for all $s \in S$ there is a unique $t \in T$ such that $(s, t) \in G$; we write $f(s)$ for this element t.

particular,

$$f(x) = 2x + 2 \quad \text{and} \quad g(x) = \frac{2x^2 - 2}{x - 1}$$

are *not* the same function, even though their values agree for all $x \neq 1$, since their domains are different.

Definition 17.1. Let S be a subset of $\mathbb{R}$. Let $f : S \to \mathbb{R}$ be a function, and a and L be real numbers. We say that *the limit of f as x approaches a is L* provided:

> for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $0 < |x-a| < \delta$, then x is in the domain of f and $|f(x) - L| < \varepsilon$.

If this happens, we write $\lim\limits_{x \to a} f(x) = L$ to denote this.

(1) Unpackaging parts of the definition.
   (a) Describe $\{x \in \mathbb{R} \mid 0 < |x - 2| < 1\}$ as a union of two open intervals.
   (b) For a general $a \in \mathbb{R}$ and $\delta > 0$, describe $\{x \in \mathbb{R} \mid 0 < |x - a| < \delta\}$ as a union of two open intervals.
   (c) Focusing on the "domain" part of the definition, if the limit of f as x approaches a is L, then f must at least be defined ________________________________ (where?).

> (a) $(1, 2) \cup (2, 3)$
> (b) $(a - \delta, a) \cup (a, a + \delta)$.
> (c) on some open intervals to the left and to the right of a.

(2) The $\varepsilon - \delta$ game.
   (a) Player 0 starts by graphing a function f (like a familiar one from calculus) and specifies an x-value a and a y-value L that (based on previous calculus knowledge) they think makes $\lim_{x \to a} f(x) = L$ **true**. [The graph should be large.]
   (b) Player 1 choses an ε. This is how close we would like our function to be to L. Thus, ε goes up and down from L (corresponding to $|f(x) - L| < \varepsilon$). Draw horizontal dotted lines with y-values $L - \varepsilon$ and $L + \varepsilon$. [The ε should be large enough for people to see and have room to work in the picture.]
   (c) Player 2 must find a δ such that every $x \in (a - \delta, a) \cup (a, a + \delta)$ is
      - in the domain of f, and
      - has an output $f(x)$ within $(L - \varepsilon, L + \varepsilon)$.

Draw vertical dotted lines for the x-values $a-\delta$ and $a+\delta$. [Everyone in the team can assist player 2!]

(d) Repeat with the same graph, players 1& 2 switching roles (and a new ε).

(3) Draw the graph of $g(x) = \dfrac{2x^2-2}{x-1}$. Play the $\varepsilon - \delta$ game with this function, $a = 1$ and $L = -3$. What happens?

So long as $\varepsilon < 7$, it is impossible for Player 2.

(4) Consider the function $g(x) = \dfrac{2x^2-2}{x-1}$. It is true that $\lim_{x\to 1} g(x) = 4$.

(a) I claim that for $\varepsilon = 3$, the choice $\delta = 1.5$ "works" to make the rest of the definition true. Verify this.

(b) Find a δ that "works" for $\varepsilon = 1$.

(c) Find a δ that "works" for $\varepsilon = 1/2$.

(d) Find a δ that "works" for $\varepsilon > 0$.

(a) Let $0 < |x-1| < 1.5$, so $-.5 < x < 2.5$ and $x \neq 1$. Then $f(x)$ is defined, since $x \neq 1$. Also, $|f(x)-4| = |2x+2-4| = |2x-2| < 2\cdot 1.5 = 3 = \varepsilon$ since $|x-1| < 1.5$.

(b) Take $\delta = .5$. Let $0 < |x-1| < .5$, so $.5 < x < 1.5$ and $x \neq 1$. Then $f(x)$ is defined, since $x \neq 1$. Also, $|f(x)-4| = |2x+2-4| = |2x-2| < 2\cdot .5 = 1 = \varepsilon$ since $|x-1| < .5$.

(c) Take $\delta = .25$. Let $0 < |x-1| < .25$, so $.75 < x < 1.25$ and $x \neq 1$. Then $f(x)$ is defined, since $x \neq 1$. Also, $|f(x)-4| = |2x+2-4| = |2x-2| < 2\cdot .25 = .5 = \varepsilon$ since $|x-1| < .25$.

(d) Take $\delta = \varepsilon/2$. Let $0 < |x-1| < \varepsilon/2$. Then $f(x)$ is defined, since $x \neq 1$. Also, $|f(x)-4| = |2x+2-4| = |2x-2| < 2\cdot \varepsilon/2 = \varepsilon$ since $|x-1| < \varepsilon/2$.

(5) Consider the function $g(x) = \dfrac{2x^2-2}{x-1}$. It is not true that $\lim_{x\to 1} g(x) = -3$. I claim that for $\varepsilon = 1$, there is no choice of $\delta > 0$ that "works" to make the rest of the definition true. Verify this.

> Let $\delta > 0$. Take $x = 1 + \delta/2$. Then $|x - 1| = \delta/2$ is between 0 and δ, and $f(x) = 2x + 2 = 4 + \delta > 4$, so $|f(x) - (-3)| = |f(x) + 3| > 7 > 1 = \varepsilon$.

18. November 1, 2022

Example 18.1. Let f be the function given by the formula

$$f(x) = \frac{5x^2 - 5}{x - 1}.$$

Recall our convention that we interpret the domain of f to be all real numbers where this rule is defined. So, $f : S \to \mathbb{R}$ where $S = \mathbb{R} \setminus \{1\}$. I claim that the limit of $f(x)$ as x approaches 1 is 10. To prove it:

Pick $\varepsilon > 0$.

(Scratch work: Since f is defined at all points other that 1, the condition about f being defined for all x such that $0 < |x - a| < \delta$ will be met for any choice of δ. We need $|f(x) - 10| < \varepsilon$ to hold. Manipulating this a bit, we see that it is equivalent to $|x - 1| < \frac{\varepsilon}{5}$. Thus setting $\delta = \frac{\varepsilon}{5}$ is the way to go. Back to the proof....)

Let $\delta = \frac{\varepsilon}{5}$. Pick x such that $0 < |x - 1| < \delta$. Then $x \neq 1$ and hence f is defined at x. We have

$$|f(x) - 10| = \left|\frac{5x^2 - 5}{x - 1} - 10\right| = \left|\frac{5x^2 - 5 - 10x + 10}{x - 1}\right| = \left|\frac{5x^2 - 10x + 5}{x - 1}\right|$$
$$= \left|\frac{5(x^2 - 2x + 1)}{x - 1}\right| = \left|\frac{5(x - 1)^2}{x - 1}\right| = |5x - 5| = 5|x - 1| < 5\delta = \varepsilon.$$

We have shown that for any $\varepsilon > 0$ there is a $\delta > 0$ such that if $0 < |x - 1| < \delta$, then f is defined at x and $|f(x) - 10| < \varepsilon$. This proves $\lim_{x\to 1} f(x) = 10$.

Example 18.2. Let's do a more complicated example: Let $f(x) = x^2$ with domain all of $\mathbb{R}$. I claim that $\lim_{x\to 2} x^2 = 4$. This is intuitively obvious but we need to prove it using just the definition.

Proof. Pick $\varepsilon > 0$.

(Scratch work: The domain of f is all of $\mathbb{R}$ and so we don't need to worry at all about whether f is defined at all. We need to figure out how small to make δ so that if $0 < |x - 2| < \delta$ then $|x^2 - 4| < \varepsilon$. The latter is equivalent to $|x-2||x+2| < \varepsilon$. We can make $|x-2|$ arbitrarily small by making δ aribitrarily small, but how can we handle $|x + 2|$? The trick is to bound it appropriately. This can be done in many ways. Certainly we can choose δ to be at most 1, so that if $|x - 2| < \delta$ then

$|x-2| < 1$ and hence $1 < x < 3$, so that $|x+2| < 5$. So, we will be allowed to assume $|x+2| < 5$. Then $|x-2||x+2| < 5|x-2|$ and $5|x-2| < \varepsilon$ provided $|x-2| < \frac{\varepsilon}{5}$. Back to the formal proof...)

Let $\delta = \min\{\frac{\varepsilon}{5}, 1\}$. Let x be any real number such that $0 < |x-2| < \delta$. Then certainly f is defined at x. Since $\delta \leq 1$ we get $|x-2| < 1$ and hence $|x+2| \leq 5$. Since $\delta \leq \frac{\varepsilon}{5}$ we have $|x-2| < \frac{\varepsilon}{5}$. Putting these together gives

$$|f(x) - 4| = |x^2 - 4| = |x-2||x+2| < |x-2|5 < \frac{\varepsilon}{5}5 = \varepsilon.$$

This proves $\lim_{x\to 2} x^2 = 4$. $\square$

Let's give an example of a function that does *not* have a limiting value as x approaches some number a.

Example 18.3. Let $f(x) = \frac{1}{x-3}$ with domain $\mathbb{R} \setminus \{3\}$. I claim that the limit of $f(x)$ as x approaches 3 does not exist. To prove this, by way of contradiction, suppose the limit of $f(x)$ as x approaches 3 does exist and is equal to L. Taking $\varepsilon = 1$ in the definition, there is a $\delta > 0$ so that if $0 < |x-3| < \delta$, then $\left|\frac{1}{x-3} - L\right| < 1$. We can find a real number x so that both $3 < x < 4$ and $0 < |x-3| < \delta$ hold. For such an x we have $\left|\frac{1}{x-3} - L\right| < 1$ and so

$$\frac{1}{x-3} - 1 < L < \frac{1}{x-3} + 1,$$

and we also have $0 < x - 3 < 1$ and so $\frac{1}{x-3} > 1$. It follows that

$$L > 0.$$

Now pick x such that $2 < x < 3$ and $0 < |x-3| < \delta$. We get

$$\frac{1}{x-3} - 1 < L < \frac{1}{x-3} + 1,$$

and $\frac{1}{x-3} < -1$ and hence

$$L < 0.$$

This is not possible; so the limit of $f(x)$ as x approaches 3 does not exist.

The following result gives an important connection between limits of functions and limits of sequences. This result will allow us to translate statements we have proven about limits of sequences to limits of functions.

Theorem 18.4. *Let $f(x)$ be a function and let a be a real number. Let $r > 0$ be a positive real number such that f is defined at every point of $\{x \in \mathbb{R} \mid 0 < |x-a| < r\}$. Let L be any real number.*

$\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a and satisfies $0 < |x_n - a| < r$ for all n, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Loosely, the condition that there is an $r > 0$ such that f is defined at every point of $\{x \in \mathbb{R} \mid 0 < |x-a| < r\}$ says that "f is defined near, but not necessarily at, a".

Proof. Let f be a function, $a \in \mathbb{R}$, and $r > 0$ a positive real number such that f is defined on $\{x \in \mathbb{R} \mid 0 < |x - a| < r\}$.

($\Rightarrow$) Assume $\lim_{x\to a} f(x) = L$. Let $\{x_n\}_{n=1}^{\infty}$ be any sequence that converges to a and is such that $0 < |x_n - a| < r$ for all n. We need to prove that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

Pick $\varepsilon > 0$. By definition of the limit of a function, there is a $\delta > 0$ such that if $0 < |x - a| < \delta$, then f is defined at x and $|f(x) - L| < \varepsilon$. Since $\delta > 0$ and $\{x_n\}_{n=1}^{\infty}$ converges to a, by the definition of convergence, there is an N such that if $n \in \mathbb{N}$ and $n > N$ then $|x_n - a| < \delta$. I claim that this N "works" to prove $\{f(x_n)\}_{n=1}^{\infty}$ converges to L too: If $n \in \mathbb{N}$ and $n > N$, then $|x_n - a| < \delta$ and, since $x_n \neq a$ for all n, we have $0 < |x_n - a| < \delta$. It follows that $|f(x_n) - L| < \varepsilon$. This shows that $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

($\Leftarrow$) We prove the contrapositive. That is, assume $\lim_{x\to a} f(x)$ is not L (including the case where the limit does not exist). We need to prove that there is at least one sequence $\{x_n\}_{n=1}^{\infty}$ such that (a) it converges to a, (b) $0 < |x_n - a| < r$ for all n and yet (c) the sequence $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L.

The fact that $\lim_{x\to a} f(x)$ is not L means:

> There is an $\varepsilon > 0$ such that for every $\delta > 0$ there exists an $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$, but either f is not defined at x or $|f(x) - L| \geq \varepsilon$.

For this ε, for any natural number n, set $\delta_n = \min\{\frac{1}{n}, r\}$. We get that there is a $x_n \in \mathbb{R}$ such that $0 < |x_n - a| < \delta_n$ and $|f(x_n) - L| \geq \varepsilon$. (Note that f is necessarily defined at x_n since $\delta_n \leq r$.) I claim that the sequence $\{x_n\}_{n=1}^{\infty}$ satisfies the needed three conditions. (a) Since $\delta_n \leq \frac{1}{n}$, we have $a - \frac{1}{n} < x_n < a + \frac{1}{n}$ for all n, and hence by the Squeeze Lemma, the sequence $\{x_n\}_{n=1}^{\infty}$ converges to a. (b) This holds by construction, since $\delta_n \leq r$. (c) Since, for the positive number ε above, we have $|f(x_n) - L| \geq \varepsilon$ for all n, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to L. □

Corollary 18.5. *Let f be a function and a and L be real numbers. Suppose that the domain of f is all of $\mathbb{R}$ or $\mathbb{R} \setminus \{a\}$. Then $\lim_{x\to a} f(x) = L$ if and only if for every sequence $\{x_n\}_{n=1}^{\infty}$ that converges to a such* *that $x_n \neq a$ for all n, we have that the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.*

Proof. ($\Rightarrow$) Assume $\lim_{x\to a} f(x) = L$, and let $\{x_n\}_{n=1}^{\infty}$ be a sequence that converges to a such that $x_n \neq a$ for all n. Since $\{x_n\}_{n=1}^{\infty}$ is convergent, it is bounded, so there is some $M > 0$ such that $|x_n| < M$ for all n. Then $|x_n - a| < M + |a|$ by the Triangle Inequality. Thus, $0 < |x_n - a| < M + |a|$ for all n, so we can apply Theorem 18.4 (with "r"$= M + |a|$), so $\{f(x_n)\}_{n=1}^{\infty}$ converges to L.

($\Leftarrow$) The point is that if the "right hand side" condition holds in this statement, then for any $r > 0$, the "right hand side" condition of Theorem 18.4 holds. Thus, by Theorem 18.4, $\lim_{x\to a} f(x) = L$. $\square$

19. November 3, 2022

Theorem 19.1 (Algebra and limits of functions)**.** *Suppose f and g are two functions and that a is a real number, and assume that*

$$\lim_{x\to a} f(x) = L \text{ and } \lim_{x\to a} g(x) = M$$

for some real numbers L and M. Then

(1) $\lim_{x\to a}(f(x) + g(x)) = L + M$.
(2) *For any real number c,* $\lim_{x\to a}(c \cdot f(x)) = c \cdot L$.
(3) $\lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M$.
(4) *If, in addition, we have that $M \neq 0$, then* $\lim_{x\to a}(f(x)/g(x)) = L/M$.

Theorem 19.2 (Squeeze Theorem for limits)**.** *Suppose f, g, and h are three functions and a is a real number. Suppose there is a positive real number $r > 0$ such that*

- *each of f, g, h is defined on $\{x \in \mathbb{R} \mid 0 < |x - a| < r\}$,*
- *$f(x) \leq g(x) \leq h(x)$ for all x such that $0 < |x - a| < r$, and*
- $\lim_{x\to a} f(x) = L = \lim_{x\to a} h(x)$ *for some number L.*

Then $\lim_{x\to a} g(x) = L$.

(1) Use the $\varepsilon - \delta$ definition to show that $\lim_{x\to 0}\ |x| = 0$.

> Let $\varepsilon > 0$. Take $\delta = \varepsilon$. Pick x such that $0 < |x| < \delta$. Then $|x|$ is defined, and $|\,|x| - 0\,| = |x| < \varepsilon$. Thus, $\lim_{x\to 0}\ |x| = 0$.

(2) Let

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$$

Use the $\varepsilon-\delta$ definition to show that $\lim_{x\to a} f(x)$ does not exist for any real number $a \in \mathbb{R}$.

Fix $a \in \mathbb{R}$ and suppose $\lim_{x\to a} f(x) = L$ for some L. Take $\varepsilon = \frac{1}{2}$. Then there is some δ such that if $0 < |x-a| < \delta$ then $|f(x) - L| < \frac{1}{2}$. By density of rationals, there is a rational number q with $a < q < a+\delta$, so $f(q) = 1$ and hence $|1-L| < \frac{1}{2}$. By density of irrationals, there is a rational number z with $a < z < a+\delta$, so $f(z) = 0$ and hence $|0-L| < \frac{1}{2}$. But then $1 = |1-0| \leq |0-L| + |1-L| < \frac{1}{2} + \frac{1}{2} = 1$, a contradiction. Thus no such L exists.

(3) Use Corollary 18.5 to show that $\displaystyle\lim_{x\to 0} \sin\left(\frac{1}{x}\right)$ does not exist.

Suggestion: Let $f(x) = \sin(\frac{1}{x})$ and suppose $\lim_{x\to 0} f(x) = L$. Find sequences $\{x_n\}_{n=1}$ and $\{y_n\}_{n=1}$ such that

- $\{x_n\}_{n=1}$ and $\{y_n\}_{n=1}$ both converge to 0,
- $f(x_n) = 1$ for all n, and
- $f(y_n) = -1$ for all n.

Suppose $\lim_{x\to 0} f(x) = L$. Let $\{x_n\}_{n=1}^{\infty} = \{\frac{1}{\frac{\pi}{2}+2\pi n}\}_{n=1}^{\infty}$. This sequence converges to 0 and $f(x_n) = 1$ for all n, so $\{f(x_n)\}_{n=1}^{\infty}$ converges to 1. Thus, $L = 1$. Now let $\{y_n\}_{n=1}^{\infty} = \{\frac{1}{\frac{-\pi}{2}+2\pi n}\}_{n=1}^{\infty}$. This sequence converges to 0 and $f(y_n) = -1$ for all n, so $\{f(y_n)\}_{n=1}^{\infty}$ converges to -1. Thus, $L = -1$. This is a contradiction, so no such L exists.

(4) Use Theorem 19.1 plus a homework problem to compute $\displaystyle\lim_{x\to 2} \frac{3x^2 - x + 2}{x+3}$.

We have $\lim_{x\to 2} x = 2$ and the limit of a constant is the value of that constant. Thus $\lim_{x\to 2} x + 3 = 2 + 3 = 5$, and $\lim_{x\to 2} x^2 = (\lim_{x\to 2} x)^2 = 4$, so $\lim_{x\to 2} 3x^2 - x + 2 = 3\cdot 4 - 2 + 2 = 12$, and hence $\displaystyle\lim_{x\to 2} \frac{3x^2 - x + 2}{x+3} = 5$.

(5) Use Theorem 19.2 to show that $\displaystyle\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$. You can use any trig facts on the bottom of the page.

We have $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$, so $-|x| \leq x\sin\left(\frac{1}{x}\right) \leq |x|$. We know that $\lim_{x\to 0} |x| = 0$ and hence $\lim_{x\to 0} -|x| = 0$ by the Theorem on algebra of limits of functions. Then by the Squeeze theorem for functions, $\lim_{x\to 0} x\sin\left(\frac{1}{x}\right) = 0$.

(6) Use Theorem 18.4 to deduce Theorem 19.2 from our Squeeze Theorem for sequences.

Proof. Let f, g, h, a, r, L be as in the statement. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence that converges to a and such that $0 < |x_n - a| < r$ for all n. By Theorem 18.4, it suffices to show that $\lim_{n\to\infty} g(x_n) = L$. By Theorem 18.4, we know that $\lim_{n\to\infty} f(x_n) = L = \lim_{n\to\infty} h(x_n)$. Since $f(x_n) \leq g(x_n) \leq h(x_n)$ for all n, we have $\lim_{n\to\infty} g(x_n) = L$ by the Squeeze Theorem (for sequences). $\square$

(7) Use Theorem 18.4 to deduce Theorem 19.1 part (1) from our Theorem 10.2 on algebra and sequences.

Proof. First, as a technical matter, we note that since we assume $\lim_{x\to a} f(x) = L$ there is a positive real number r_1 such that $f(x)$ is defined for all x satisfying $0 < |x-a| < r_1$, and likewise since $\lim_{x\to a} g(x) = M$ there is a positive real number r_2 such that $g(x)$ is defined for all x satisfying $0 < |x - a| < r_2$. Letting $r = \min\{r_1, r_2\}$, we have that $r > 0$ and $f(x)$ and $g(x)$ and hence $f(x) + g(x)$ are defined for all x satisfying $0 < |x - a| < r$. (We needed to prove this in order to apply Theorem 18.4.)

Let $\{x_n\}_{n=1}^{\infty}$ be any sequence converging to a such that $0 < |x_n - a| < r$ for all n. By Theorem 18.4 in the "forward direction", we have that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. By Theorem 10.2, $\lim_{n\to\infty} f(x_n) + g(x_n) = L + M$. So, by Theorem 18.4 again (this time applying it to $f(x) + g(x)$ and using the "backward implication"), it follows that $\lim_{x\to a}(f(x) + g(x)) = L + M$. $\square$

(8) Use Theorem 18.4 to deduce Theorem 19.1 part (4) from our Theorem 10.2 on algebra and sequences.

Proof. Since we assume $\lim_{x\to a} f(x) = L$ there is a positive real number r_1 such that $f(x)$ is defined for all x satisfying $0 < |x - a| < r_1$. Since $\lim_{x\to a} g(x) = M$ there is a positive real number r_2 such that $g(x)$ is defined for all x satisfying $0 < |x - a| < r_2$. Since $M \neq 0$, $|M| > 0$, and applying definition of limit, there is some $\delta > 0$ such that if $0 < |x - a| < \delta$, then $|g(x) - M| < |M|$, and hence by the reverse triangle inequality, $|g(x)| \geq ||M| - |g(x) - M|| > 0$, so $g(x) \neq 0$.

Letting $r = \min\{r_1, r_2, \delta\}$, we have that $r > 0$ and $f(x)$, $g(x)$, $1/g(x)$, and hence $f(x)/g(x)$ are defined for all x satisfying $0 < |x - a| < r$.

Let $\{x_n\}_{n=1}^{\infty}$ be any sequence converging to a such that $0 < |x_n - a| < r$ for all n. By Theorem 18.4 in the "forward direction", we have that $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} g(x_n) = M$. Since $M \neq 0$ and $g(x_n) \neq 0$ for all $n \in \mathbb{N}$, by Theorem 10.2,
$\lim_{n\to\infty} f(x_n)/g(x_n) = L/M$. So, by Theorem 18.4 again, it follows that $\lim_{x\to a}(f(x)/g(x)) = L/M$. $\square$

20. November 8, 2022

We come to the formal definition of continuity. We first define what it means for a function to be continuous *at a single point*, but ultimately we will be interested in functions that are continuous on entire intervals.

Definition 20.1. Suppose f is a function and a is a real number. We say *f is continuous at a* provided the following condition is met:

> For every $\varepsilon > 0$ there is a $\delta > 0$ such that if x is a real number such that $|x - a| < \delta$ then f is defined at x and $|f(x) - f(a)| < \varepsilon$.

Remark 20.2. If f is continuous at a, then by applying the definition using any positive number $\varepsilon > 0$ you like (e.g., $\varepsilon = 1$) we get that there exists a $\delta > 0$ such that f is defined for all x such that $a - \delta < x < a + \delta$. That is, in order for f to be continuous at a it is necessary (but not sufficient) that f is defined at all points near *a including at a itself*. In particular, unlike in the definition of "limit", f must be defined at a in order for it to possibly be continuous at a.

Example 20.3. I claim $f(x) = 3x$ is continuous at a for every value of a. Pick $\varepsilon > 0$. Let $\delta = \frac{\varepsilon}{3}$. If $|x - a| < \delta$ then f is defined at x (since the domain of f is all of $\mathbb{R}$) and

$$|f(x) - f(a)| = |3x - 3a| = 3|x - a| < 3\delta = \varepsilon.$$

Example 20.4. The function $f(x)$ with domain $\mathbb{R}$ defined by

$$f(x) = \begin{cases} 2x - 7 & \text{if } x \geq 3 \text{ and} \\ -x & \text{if } x < 3 \end{cases}$$

is not continuous at 3. Since the domain of f is all of $\mathbb{R}$, the negation of the definition of "continuous at 3" is:

> there is an $\varepsilon > 0$ such that for every $\delta > 0$ there is a real number x such that $|x - 3| < \delta$ and $|f(x) - f(3)| \geq \varepsilon$.

Set $\varepsilon = 1$. For any $\delta > 0$, we may choose a real number x so that $3 - \delta < x < 3$ and $2.9 < x < 3$. For such an x, we have

$$|f(x) - f(3)| = |-x + 1| = x - 1 > 1.9 > \varepsilon.$$

The proves f is not continuous at 3.

The definition of continuous looks a lot like the definition of limit, with L replaced by $f(a)$. This is not just superficial:

Theorem 20.5. *Suppose f is a function and a is a real number and assume that f is defined at a. f is continuous at a if and only if $\lim_{x\to a} f(x) = f(a)$.*

Proof. ($\Rightarrow$) This is immediate from the definitions.

($\Leftarrow$) This is almost immediate from the definitions: Suppose $\lim_{x\to a} f(x) = f(a)$. Pick $\varepsilon > 0$. Then there is a δ such that if $0 < |x - a| < \delta$, then f is defined at x and $|f(x) - f(a)| < \varepsilon$. This nearly gives that f is continuous at a by definition, except that we need to know that if $|x - a| < \delta$, then f is defined at x and $|f(x) - f(a)| < \varepsilon$. The only "extra" case is the case $x = a$. But if $x = a$, then f is defined at a by assumption and we have $|f(x) - f(a)| = 0 < \varepsilon$. $\square$

Remark 20.6. Remember, when we write $\lim_{x\to a} f(x) = f(a)$ we mean that the limit exists and is equal to the number $f(a)$. So, by this Lemma, if $\lim_{x\to a} f(x)$ does not exist, then f is not continuous at a.

Example 20.7. Define a function f whose domain is all of $\mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

As we proved, $\lim_{x\to a} f(x)$ does not exist for any a. So, this function is continuous nowhere.

Example 20.8. The function $f(x) = \sqrt{x}$ is continuous at a for every $a > 0$. This holds since for any $a > 0$, as you proved on the homework we have

$$\lim_{x \to a} \sqrt{x} = \sqrt{a}.$$

Theorem 20.9. *Let $a \in \mathbb{R}$ and suppose f and g are two functions that are both continuous at a. Then so are*

(1) $f(x) + g(x)$,
(2) $c \cdot f(x)$, *for any constant* c,
(3) $f(x) \cdot g(x)$, *and*
(4) $\frac{f(x)}{g(x)}$ *provided* $g(a) \neq 0$.

Proof. Follows from Theorems 20.5 and 19.1. □

Example 20.10. Polynomials are continuous everywhere. The function x is continuous everywhere (since $\lim_{x \to a} x = a$). By part (3) above and a simple induction, x^n is continuous everywhere for every n. Then by parts (1) and (2), it follows that every polynomial is continuous everywhere.

Recall that for functions f and g, $f \circ g$ is the *composition*: it is the function that sends x to $f(g(x))$. The domain of $f \circ g$ is

$$\{x \in \mathbb{R} \mid \ x \text{ is the domain of } g \text{ and } g(x) \text{ is in the domain of } f\}.$$

Theorem 20.11. *Suppose g is continuous at a point a and f is continuous at $g(a)$. Then $f \circ g$ is continuous at a.*

Proof. Let $a \in \mathbb{R}$ be such that that g is continuous at a and f is continuous at $g(a)$. I prove $f \circ g$ is continuous at a using the definition.

Pick $\varepsilon > 0$. Since f is continuous at $g(a)$, there is a $\gamma > 0$ such that if $|y - g(a)| < \gamma$ then f is defined at y and $|f(y) - f(g(a))| < \varepsilon$. (I am using y in place of the usual x for clarity below, and I am calling this number γ, and not δ, since it is not the δ I am seeking.) Since $\gamma > 0$ and g is continuous at a, there is a $\delta > 0$ such that if $|x - a| < \delta$ then g is defined at x and $|g(x) - a| < \gamma$.

This δ "works" to prove $f \circ g$ is continuous at a: Let x be any real number such that $|x - a| < \delta$. Then g is defined at x and $|g(x) - g(a)| < \gamma$. Taking $y = g(x)$ above, this gives that f is defined at $g(x)$ and $|f(g(x)) - f(g(a))| < \varepsilon$. This proves $f \circ g$ is continuous at a. □

21. November 10, 2022

(1) Let
$$f(x) = \begin{cases} 2x & \text{if } x \geq 1 \\ x+1 & \text{if } x < 1 \end{cases}.$$
Use the $\varepsilon - \delta$ definition to show that $f(x)$ is continuous at 1.

> Let $\varepsilon > 0$. Take $\delta = \varepsilon/2$. Then if $|x-1| < \delta = \varepsilon/2$, we either have $f(x) = 2x$, so $|f(x) - f(1)| = |2x-2| = 2|x-1| < 2\varepsilon/2 = \varepsilon$, or $f(x) = x+1$, so $|f(x) - f(1)| = |x+1-2| = |x-1| < \varepsilon/2 < \varepsilon$.

(2) Let
$$g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}.$$
Show that $g(x)$ is continuous at 0 and is *not* continuous at any other real number. You can use any theorems you like and anything relevant from the homework.

> From the homework, $\lim_{x\to 0} f(x) = 0$, and since $f(0) = 0$, f is continuous at 0. Also from the homework, $\lim_{x\to a} f(x)$ does not exist for $a \neq 0$, so f is not continuous at a if $a \neq 0$.

(3) Let $h(x) = \sqrt{x^2+5}$. Show that h is continuous at a for every $a \in \mathbb{R}$.

> We can write $h = f \circ g$ with $f(x) = \sqrt{x}$ and $g(x) = x^2+5$. For any $a \in \mathbb{R}$, g is continuous at a, and $g(a) > 0$. Then since $g(a) > 0$, f is continuous at $g(a)$. Thus, $f \circ g$ is continuous at a.

It is tiresome to say "continuous at a for every $a \in \mathbb{R}$". The following definition is then convenient.

Definition 21.1. Let S be an open interval of $\mathbb{R}$ of the form $S = (a,b)$, $S = (a,\infty)$, $S = (-\infty,a)$, or $S = (-\infty,\infty) = \mathbb{R}$. We say f is *continuous on* S if f is continuous at a for all $a \in S$.

(4) Which of the following functions are continuous on $\mathbb{R}$?

- $f(x) = \sqrt{x^2+5}$.
- Every polynomial function.
- $f(x) = \sqrt{x}$.
- $f(x) = \frac{1}{x}$.

Just $f(x) = \sqrt{x^2+5}$ and Every polynomial function.

(5) Which of the following functions are continuous on $(0, \infty)$?

- $f(x) = \sqrt{x^2+5}$.
- Every polynomial function.
- $f(x) = \sqrt{x}$.
- $f(x) = \frac{1}{x}$.

All of them.

(6) Prove that $j(x) = \begin{cases} x\sin(1/x) \text{if}\, x \neq 0 \\ 0 \text{if}\, x = 0 \end{cases}$ is continuous on $\mathbb{R}$. (You can use without proof that $\sin(x)$ is continuous on $\mathbb{R}$).

We saw earlier that $\lim_{x\to 0} j(x) = 0 = j(0)$, so j is continuous at 0. For $a \neq 0$, the function $1/x$ is continuous at a, and $\sin(x)$ is continuous at $1/a$, so $\sin(1/x)$ is continuous at a. The function x is also continuous at a, so $x\sin(1/x)$ is continuous at a.

(7) Prove or disprove: If f and g are two functions, $a \in \mathbb{R}$, and $f(a) = g(a)$, then f is continuous at a if and only if g is continuous at a.

To disprove it, consider $f(x) = 0$ and $g(x) = \begin{cases} 0 \text{ if } x = 0 \\ 1 \text{ if } x \neq 0 \end{cases}$.

(8) Prove or disprove: If f and g are two functions, $a < b$, and $f(x) = g(x)$ for all $x \in (a, b)$, then f is continuous on (a, b) if and only if g is continuous on (a, b).

To prove it, let f be continuous on (a, b), $c \in (a, b)$, and $\varepsilon > 0$. By definition of continuous at c, there is some $\delta_1 > 0$

such that if $|x-c|<\delta$ then $|f(x)-f(c)|<\varepsilon$. Let $\delta = \min\{\delta_1, c-a, b-c\}$. Since δ is the minimum of three positive numbers, $\delta > 0$. Then, if $|x-c|<\delta$, since $|x-c|<c-a$, we must have $c>a$; since $|x-c|<b-c$, we must have $c<b$. Thus $x \in (a,b)$, so $g(x)=f(x)$. Since $|x-c|<\delta_1$, $|f(x)-f(c)|<\varepsilon$. Then $|g(x)-g(c)| = |f(x)-f(c)| < \varepsilon$. This shows that g is continuous at c. Since $c \in (a,b)$ was arbitrary, g is continuous on (a,b).

The other implication follows by switching the roles of f and g.

22. November 22, 2022

Definition 22.1. Given a function $f(x)$ and real numbers $a<b$, we say f is *continuous on the closed interval* $[a,b]$ provided

(1) for every $r \in (a,b)$, f is continuous at r in the sense defined already,
(2) for every $\varepsilon > 0$ there is a $\delta > 0$ such that $a \le x < a+\delta$, then $f(x)$ is defined and $|f(x)-f(a)|<\varepsilon$.
(3) for every $\varepsilon > 0$ there is a $\delta > 0$ such that if $b-\delta < x \le b$, then $f(x)$ is defined and $|f(x)-f(b)|<\varepsilon$.

(1) Explain why if f is continuous at x for every $x \in [a,b]$, then f is continuous on the closed interval $[a,b]$. In particular, if f is continuous on any open interval containing $[a,b]$, then f is continuous on $[a,b]$. Conclude that every polynomial is continuous on every closed interval.

Condition (1) is automatic. If f is continuous at a then there is a δ such that if $a-\delta<x<a+\delta$ then $f(x)$ is defined and $|f(x)-f(a)|<\varepsilon$; in particular, for the same δ, if $a \le x < a+\delta$, then $a-\delta<x<a+\delta$ and hence $f(x)$ is defined and $|f(x)-f(a)|<\varepsilon$. Similarly for continuity at b and condition (3).

(2) Show that the function $f(x) = \sqrt{1-x^2}$ is continuous on the closed interval $[-1,1]$:
- For showing condition (1), I recommend using a Theorem about compositions of functions.
- For conditions (2) and (3), show that $\delta = \min\{\varepsilon^2/\sqrt{4}, 2\}$ works.

Is this function continuous on any open interval containing $[-1, 1]$?

If $x \in (-1, 1)$, then write $f(x) = (g \circ h)(x)$ with $h(x) = 1 - x^2$ and $g(x) = \sqrt{x}$; for x in this range, $h(x) > 0$, so g is continuous at x, and hence f is continuous at x. This covers condition (1). For condition (2), let $\varepsilon > 0$ and take $\delta = \min\{\varepsilon^2/4, 2\}$. If $-1 \le x < -1 + \delta$, then since $\delta \le 2$, $x < -1 + 2 = 1$, so x is in the domain of f. Also, since $\delta \le \varepsilon^2/4$, we have $x - (-1) < \delta \le \varepsilon^2/4$ and since $x \ge -1$, $1 - x \le 2$, so

$$|\sqrt{1-x^2} - \sqrt{1-(-1)^2}| = \sqrt{1-x^2} = \sqrt{1-x}\sqrt{1+x} < 2\sqrt{\varepsilon^2/4} = \varepsilon.$$

Similarly for condition (3).

Theorem 22.2 (Intermediate Value Theorem)**.** *Suppose f is a function, that $a < b$ are real numbers, and that f is continuous on the closed interval $[a, b]$. If y is any number between $f(a)$ and $f(b)$ (i.e., $f(a) \le y \le f(b)$ or $f(a) \ge y \ge f(b)$), then there is a $c \in [a, b]$ such that $f(c) = y$.*

(3) Draw a picture of this theorem as follows:
- Mark some a and b on the x-axis.
- Graph a function f that is continuous on $[a, b]$.
- Mark $f(a)$ and $f(b)$ on the y-axis.
- Pick some y in between $f(a)$ and $f(b)$, and make a horizontal line for this y-value.
- Does it intersect the graph of f?

Repeat with at least one graph that is increasing, at least one graph that is decreasing, and at least one graph that is neither increasing nor decreasing.

(4) Give a counterexample to the statement of the Intermediate Value Theorem without the hypothesis that f is continuous on $[a, b]$.

Take

$$f(x) = \begin{cases} x \text{ if } x < 0 \\ x + 2 \text{ if } x \ge 0 \end{cases}.$$

Then $f(-1) < 1 < f(1)$, but there is no $x \in [-1, 1]$ with $f(x) = 1$.

(5) Prove or disprove: There is a real number $x \in [0, 2]$ such that $x^3 - 3x = 1$.

Take $f(x) = x^3 - 3x$. It is a polynomial and hence continuous on $[0, 2]$. Then $f(0) = 0 < 1 < 2 = f(2)$, so there is some $c \in [-1, 1]$ with $f(c) = 1$.

(6) Prove or disprove: There are at least two real numbers $x \in [0, 2]$ such that $x^3 - 3x = -1$.

Take the same $f(x) = x^3 - 3x$. Then $f(0) = 0 > -1 > -2 = f(1)$, so there is some $c_1 \in [0, 1]$ with $f(c_1) = 1$. Also, Then $f(1) = -2 < -1 < 2 = f(2)$, so there is some $c_2 \in [1, 2]$ with $f(c_2) = 1$. We know that $c_1 \neq 1$ and $c_2 \neq 1$ since $f(1) \neq 1$, so $c_1 \neq c_2$. Thus there are two values that output -1.

(7) True or false: If $f(x)$ is continuous on $[a, b]$, and y is *not* in between $f(a)$ and $f(b)$, then there is no $c \in [a, b]$ such that $f(c) = y$.

From the previous problem $f(x) = x^3 - 3x$ on $[0, 2]$ with $y = -1$ is a counterexample.

(8) **Proof of the Intermediate Value Theorem:**
  (a) Let's assume that $f(a) \leq f(b)$ to get started. Explain why the cases $y = f(a)$ and $y = f(b)$ are easy. Hence, we assume that $f(a) < y < f(b)$.
  (b) Let $S = \{x \in [a, b] \mid f(r) < y \text{ for all } a \leq r \leq x\}$. In short, S is the set of x-values in the interval where the graph of f hasn't crossed y yet. Explain why S has a supremum, and let $c = \sup(S)$.
  (c) Show that $c > a$. [Hint: Apply part (2) of definition of continuous on $[a, b]$ with $\varepsilon = y - f(a)$, and show that a is not an upper bound for S.]

(d) The argument that $c < b$ is similar (so come back to it later if you want). Thus, $c \in (a, b)$, so we know that f is continuous at c.

(e) Suppose that $f(c) < y$, and obtain a contradiction. [Hint: Apply continuous at c with $\varepsilon = y - f(c)$, and show that c is not an upper bound for S.]

(f) Suppose that $f(c) > y$, and obtain a contradiction. [Hint: Apply continuous at c with $\varepsilon = f(c) - y$, and find a smaller upper bound for S.]

(g) This concludes the case when $f(a) \leq f(b)$. If $f(a) \geq f(b)$, what can you say about $g(x) = -f(x)$? Can we apply the case we just did?

Proof of Intermediate Value Theorem. Assume f is continuous on $[a, b]$ and y is a real number such that $f(a) \leq y \leq f(b)$ or $f(b) \leq y \leq f(a)$. We need to prove there is a $c \in [a, b]$ such that $f(c) = y$.

Let us assume $f(a) \leq y \leq f(b)$ — the other case may be proved in a very similar manner, or by appealing to this case using the function $-f(x)$ instead.

If $f(a) = y$ then we may take $c = a$ and if $f(b) = y$ then we may take $c = b$. So, we may assume $f(a) < y < f(b)$.

Consider the set

$$S = \{z \in \mathbb{R} \mid a \leq z \leq b \text{ and } f(x) < y \text{ for all } x \in [a, z]\}$$

This set is nonempty, since $a \in S$, and it is bounded above, by b. It therefore has a supremum, which we will call c. I claim $f(c) = y$.

Let us first show that $c > a$. By way of contradiction, suppose $c \leq a$. Since $c \geq a$, we must have $c = a$. Since f is continuous on $[a, b]$, taking $\varepsilon = y - f(a) > 0$ in the definition, we get that there is a $\delta > 0$ such that if $a \leq x < a + \delta$, then $f(a) - \varepsilon < f(x) < f(a) + \varepsilon$. In particular, if $a \leq x \leq a + \delta/2$, then $f(x) < f(a) + \varepsilon = y$. This proves that $a + \delta/2 \in S$. But $a + \delta/2 > a = c$, contrary to the fact that c is the supremum of S. We conclude that $c > a$.

Similarly, one may show that $c < b$ — I leave this to you as an exercise.

We now know that $a < c < b$, and we next prove that $f(c) = y$ by showing that $f(c) > y$ and $f(c) < y$ are each impossible.

Suppose $f(c) > y$. Setting $\varepsilon = f(c) - y$ and applying the definition of continuous at c, there is a $\delta > 0$ such that if x is any number such that $c - \delta < x < c + \delta$ then $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. In particular, for any z such that $c - \delta < z \leq c$, we have

$$f(z) > f(c) - \varepsilon = y.$$

In particular, z is not in the set S. It follows that $c - \delta$ is an upper bound of S, contrary to the fact that c is the least upper bound of S.

Suppose $f(c) < y$. Setting $\varepsilon = y - f(c)$ and applying the definition of continuous at c, there is a $\delta > 0$ such that if $c - \delta < x < c + \delta$, then $f(c) - \varepsilon < f(x) < f(c) + \varepsilon$. In particular, if x is any real number such that $c \leq x \leq c + \delta/2$, then $f(x) < f(c) + \varepsilon = y$. Moreover, if $x < c$, then x is not an upper bound of S, and hence there is a $z \in S$ such that $x < z$. If follows that $f(x) \leq y$. So, we have shown that if $x \leq c + \delta/2$, then $f(x) < y$. This shows that $c + \delta/2 \in S$, contrary to c being an upper bound of S. $\square$

23. November 29, 2022

Theorem 23.1 (Boundedness Theorem)**.** *Suppose f is continuous on the closed interval $[a, b]$ for some real numbers a, b with $a < b$. Then f is bounded on $[a, b]$ — that is, there are real numbers m and M so that $m \leq f(x) \leq M$ for all $x \in [a, b]$.*

Theorem 23.2 (Extreme Value Theorem)**.** *Assume f is continuous on the closed interval $[a, b]$ for some real numbers a and b with $a < b$. Then f attains a minimum and a maximum value on $[a, b]$ — that is, there exists a number $r \in [a, b]$ such that $f(x) \leq f(r)$ for all $x \in [a, b]$ and there exists a number $s \in [a, b]$ such that $f(x) \geq f(s)$ for all $x \in [a, b]$.*

(1) Explain why the Extreme Value Theorem actually implies the Boundedness Theorem. (The reason we state both is that we have to prove the Boundedness Theorem on the way to the Extreme Value Theorem.)

Take $M = f(r)$ and $m = f(s)$.

(2) In this problem we explore the necessity of the hypotheses in these theorems.
  (a) Draw a graph of a function on a closed interval $[a, b]$ that is *not continuous*, but is not bounded on $[a, b]$.
  (b) Draw a graph of a function that is continuous on an *open* interval (a, b), but is not bounded on (a, b).
  (c) Draw a graph of a function that is continuous on an *open* interval (a, b), and is bounded on (a, b), but for which the conclusion of the Extreme Value Theorem fails.

  Can you find formulas of functions that match each story?

(a) For example, take $f(x) = \begin{cases} 1/x \text{ if } x \neq 0 \\ 0 \text{ if } x = 0 \end{cases}$ on $[-1, 1]$.
(b) For example, take $f(x) = 1/x$ on $(0, 1)$.
(c) For example, take $f(x) = x$ on $(0, 1)$.

Lemma from homework: Let $a < b$ be real numbers and $[a, b]$ be a closed interval. Let $\{x_n\}_{n=1}$ be a sequence with $x_n \in [a, b]$ for all n, and assume that $\{x_n\}_{n=1}$ converges to r. Then,

- $r \in [a, b]$, and
- If f is continuous on the closed interval $[a, b]$, then the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(r)$.

(3) Proof of Boundedness Theorem:
  (a) We argue by contradiction. What does it mean to suppose that the theorem is false? Your answer should involve an "or". Assume one of the two cases.
  (b) Explain why there must be a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [a, b]$ and $f(x_n) > n$ for all $n \in \mathbb{N}$ (unless you chose the other case. . .).
  (c) Apply Bolzano-Weierstrass to the sequence $\{x_n\}_{n=1}^{\infty}$. What do you get?
  (d) Now apply the Lemma from the homework. What do you get?

(a) If the Theorem is false, there is a function f that is continuous on the closed interval $[a, b]$ but the set of values of f on $[a, b]$ is either unbounded above or unbounded below. Assume we have such a function that is unbounded above.

(b) By assumption, for every $n \in \mathbb{N}$, n is not an upper bound for the values of f on $[a, b]$. Thus, for every $n \in \mathbb{N}$, there is some x_n with $f(x_n) > n$. This gives us a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [a, b]$ and $f(x_n) > n$ for all $n \in \mathbb{N}$

(c) The sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.

(d) Since $x_{n_k} \in [a, b]$ for all k, by the lemma we have first that $\{x_{n_k}\}_{k=1}^{\infty}$ converges to $r \in [a, b]$, and second, that $\{f(x_{n_k})\}_{k=1}^{\infty}$ converges to $f(r)$. But since $f(x_{n_k}) > n_k \geq k$, the sequence $\{f(x_{n_k})\}_{k=1}^{\infty}$ diverges (to $+\infty$ in fact). This is a contradiction, so we conclude that the set of values of f is bounded above. A similar argument (or the same case applied to $-f$) shows that the set of values of f is also bounded below.

(4) Proof of Extreme Value Theorem:

We will find a maximum value; finding a minimum value is similar (or follows from this part applied to $-f$).

(a) Let $R = \{f(x) \mid x \in [a, b]\}$. Explain why R has a supremum; call it ℓ.

(b) Explain why there must be a sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in [a, b]$ and $\ell - \frac{1}{n} < f(x_n) \leq \ell$ for all $n \in \mathbb{N}$.

(c) Apply Bolzano-Weierstrass to the sequence $\{x_n\}_{n=1}^{\infty}$. What do you get?

(d) Now apply the Lemma from the homework. What do you get?

We will find a maximum value; finding a minimum value is similar (or follows from this part applied to $-f$).

(a) Let $R = \{f(x) \mid x \in [a, b]\}$. R is nonempty since $f(a) \in R$, and it is bounded above by the Boundedness Theorem. Thus R has a supremum, which we will call ℓ.

(b) For any n, $\ell - \frac{1}{n} < \sup(R)$, so it is not an upper bound for R. This means that there is some $y_n \in R$ with $\ell - \frac{1}{n} < y_n$; since ℓ is an upper bound for R we also have $\ell - \frac{1}{n} < y_n \leq \ell$. By definition of R, there is some $x_n \in [a, b]$ with $f(x_n) = y_n$. Thus, we have a sequence

$\{x_n\}_{n=1}^{\infty}$ with $x_n \in [a,b]$ and $\ell - \frac{1}{n} < f(x_n) \le \ell$ for all $n \in \mathbb{N}$.

(c) The sequence $\{x_n\}_{n=1}^{\infty}$ has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$.

(d) Since $x_{n_k} \in [a,b]$ for all k, by the lemma we have first that $\{x_{n_k}\}_{k=1}^{\infty}$ converges to $r \in [a,b]$, and second, that $\{f(x_{n_k})\}_{k=1}^{\infty}$ converges to $f(r)$. Since $\ell - \frac{1}{n_k} < f(x_{n_k}) \le \ell$, the sequence $\{f(x_{n_k})\}_{k=1}^{\infty}$ converges to ℓ by the Squeeze Theorem. Thus, $f(r) = \ell$. Then, by definition of supremum, we have that $f(x) \le f(r)$ for all $x \in [a,b]$.

24. December 1, 2022

Definition: Let f be a function and r be a real number. We say that f is *differentiable at* r if f is defined at r and the limit

$$\lim_{x \to r} \frac{f(x) - f(r)}{x - r}$$

exists. In this case, we call the limit *the derivative of* f *at* r and write $f'(r)$ for this limit.

(1) Use the definition to show that the derivative of $f(x) = x$ is 1 for any r.

Let $f(x) = x$ and consider

$$\lim_{x \to r} \frac{f(x) - f(r)}{x - r} = \lim_{x \to r} \frac{x - r}{x - r} = \lim_{x \to r} 1 = 1.$$

(2) Use the definition to show that the function $f(x) = |x|$ is *not* differentiable at $x = 0$.

Consider

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{|x|}{x} = \lim_{x \to 0} \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Letting $x_n = 1/n$, we have that the sequence $x_n \to 0$ and $f(x_n) \to 1$; letting $y_n = -1/n$, we have that the sequence $y_n \to 0$ and $f(y_n) \to -1$, so the limit does not exist.

(3) Prove that if f is differentiable at $x = r$, then f is continuous at $x = r$.

> Suppose that f is differentiable at r so
> $$f'(r) = \lim_{x \to r} \frac{f(x) - f(r)}{x - r}.$$
> Then
> $$\lim_{x \to r}(x - r)\frac{f(x) - f(r)}{x - r} = (\lim_{x \to r}(x - r))f'(r) = 0,$$
> so
> $$\lim_{x \to r} f(x) - f(r) = \lim_{x \to r}(x - r)\frac{f(x) - f(r)}{x - r} = 0,$$
> and hence
> $$\lim_{x \to r} f(x) = f(r).$$
> Thus f is continuous at r.

(4) Prove or disprove the converse of the previous statement.

> We have already seen that $f(x) = |x|$ at $r = 0$ is a counterexample.

Theorem (Derivatives and algebra: Let f, g be functions that are differentiable at $x = r$, and c be a real number. Then,

(1) $f + g$ is differentiable at $x = r$ and $(f + g)'(r) = f'(r) + g'(r)$;
(2) cf is differentiable at $x = r$ and $(cf)'(r) = cf'(r)$;
(3) fg is differentiable at $x = r$ and $(fg)'(r) = f'(r)g(r) + f(r)g'(r)$.

(6) Prove that if $f(x) = x^n$, then f is differentiable at any value of x and $f'(x) = nx^{n-1}$ for every $n \in \mathbb{N}$.

> We proceed by induction on n. For the base case, we use the definition to show that the derivative of x is 1 (skipped here). Suppose that $(x^k)' = kx^{k-1}$. Then $(x^{k+1})' = x(x^k)' + x'(x^k) = xkx^{k-1} + x^k = (k + 1)x^k$. This shows the claim for all $n \in \mathbb{N}$ by induction.

(7) Use the Theorem plus the previous problem to compute the derivative of $f(x) = 5x^7 - \sqrt{19}x^4$.

$35x^6 - 4\sqrt{19}x^3.$

(8) Prove the Theorem.

For part (1), we note that

$$\frac{(f+g)(x)-(f+g)(r)}{x-r} = \frac{f(x)+g(x)-f(r)-g(r)}{x-r} = \frac{f(x)-f(r)}{x-r} + \frac{g(x)-g(r)}{x-r}.$$

When we take the limit as x approaches r, this is $f'(r) + g'(r)$, using the definition of $f'(r)$ and $g'(r)$ and the fact that the limit of a sum of two functions is the sum of the limits (when they both exist).

For (2), we note that

$$\frac{(cf)(x)-(cf)(r)}{x-r} = c\frac{f(x)-f(r)}{x-r},$$

and it follows from our limit theorems that the limit as x approaches r is $cf'(r)$.

For (3), using what we know about limits we get

$$\lim_{x\to r} \frac{f(x)g(x)-f(r)g(r)}{x-r} = \lim_{x\to r} \left(\frac{f(x)g(x)-f(r)g(x)}{x-r} + \frac{f(r)g(x)-f(r)g(r)}{x-r} \right)$$
$$= \lim_{x\to r} g(x) \cdot \lim_{x\to r} \left(\frac{f(x)-f(r)}{x-r} \right) + f(r) \cdot \lim_{x\to r} \left(\frac{g(x)-g(r)}{x-r} \right)$$
$$= g(r)f'(r) + f(r)g'(r),$$

where for the last step we use that $\lim_{x\to r} g(x) = g(r)$ since g is continuous at r (since differentiable implies continuous).

Derivatives and optimization

Theorem: Let f be a function that is differentiable at $x = r$.

(1) If $f'(r) > 0$, then there is some $\delta > 0$ such that
- if $x \in (r, r+\delta)$ then $f(r) < f(x)$;
- if $x \in (r-\delta, r)$ then $f(x) < f(r)$.

(2) If $f'(r) < 0$, then there is some $\delta > 0$ such that
- if $x \in (r, r+\delta)$ then $f(r) > f(x)$;
- if $x \in (r-\delta, r)$ then $f(x) > f(r)$.

Corollary (Derivatives and optimization): Let f be a function that is continuous on a closed interval $[a, b]$. If f attains a maximum or minimum value on $[a, b]$ at $r \in (a, b)$, and f is differentiable at r, then $f'(r) = 0$.

(1) Find the values of x on $[0, 2]$ at which $f(x) = x^3 - 2x$ achieves its minimum and maximum values.

Since f is differentiable on $(0, 2)$, they must be where the derivative is zero or at the endpoints. $f'(x) = 0$ means $3x^2 = 6$ so $x = \pm\sqrt{2}$. The minimum is at $x = \sqrt{2}$ and the max is at $x = 2$.

(2) Explain why the Corollary follows from the Theorem.

Suppose f attains a max on $[a, b]$ at $r \in (a, b)$ and f is differentiable at r, but $f'(r) \neq 0$. If $f'(r) > 0$, then the first bullet of part (1) gives an x that yields a contradiction; if $f'(r) < 0$, then the second bullet bullet of part (2) gives an x that yields a contradiction. If f attains a min, similarly with the other bullet points.

25. December 6, 2022

Last time we considered the following theorem about derivatives and nearby values:

Theorem 25.1. *Let f be a function that is differentiable at $x = r$.*

(1) If $f'(r) > 0$, then there is some $\delta > 0$ such that
- *if $x \in (r, r + \delta)$ then $f(r) < f(x)$;*
- *if $x \in (r - \delta, r)$ then $f(x) < f(r)$.*

(2) If $f'(r) < 0$, then there is some $\delta > 0$ such that
- *if $x \in (r, r + \delta)$ then $f(r) > f(x)$;*
- *if $x \in (r - \delta, r)$ then $f(x) > f(r)$.*

Proof. We start with part (1). Take $h(x) = \frac{f(x)-f(r)}{x-r}$. Let

$$\varepsilon = f'(r) = \lim_{x \to r} h(x).$$

Then there is some $\delta > 0$ such that $|h(x) - f'(r)| > \varepsilon$ when $x \in (r - \delta, r) \cup (r, r + \delta)$. But $|h(x) - f'(r)| < f'(r)$ implies $h(x) > 0$. If $h(x) > 0$ and $x > r$ then $x - r > 0$ so $f(x) - f(r) = h(x)(x - r) > 0$.

Thus, if $x \in (r, r+\delta)$, then $f(x) > f(r)$. If $x < r$, then $x - r < 0$ so $f(x) - f(r) = h(x)(x-r) < 0$. Thus, if $x \in (r-\delta, r)$, then $f(x) < f(r)$. Part (2) is similar. $\square$

We now want to use derivatives to study when a function is increasing or decreasing.

Definition 25.2. Let f be a function, and $I \subseteq \mathbb{R}$ be an interval contained in domain of f. We say that

- f is *increasing* on I if for any $a, b \in I$ with $a < b$ we have $f(a) \leq f(b)$;
- f is *decreasing* on I if for any $a, b \in I$ with $a < b$ we have $f(a) \geq f(b)$;
- f is *constant* on I if for any $a, b \in I$ with $a < b$ we have $f(a) = f(b)$;
- f is *strictly increasing* on I if for any $a, b \in I$ with $a < b$ we have $f(a) < f(b)$;
- f is *strictly decreasing* on I if for any $a, b \in I$ with $a < b$ we have $f(a) > f(b)$.

We may be tempted by the previous theorem to say that if the derivative of f is positive then f is increasing. Before we get ahead of ourselves, let's consider a strange example.

Example 25.3. Consider the function $f(x) = \begin{cases} x^2 + x & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \notin \mathbb{Q} \end{cases}$. Note that we can write $f(x) = f_1(x) + f_2(x)$ where $f_1(x) = x$ and $f_2(x) = \begin{cases} x^2 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$. We have shown that f_1 is differentiable at any $x \in \mathbb{R}$ and $f_1'(x) = 1$, and f_2 is differentiable at $x = 0$ with $f_2'(0) = 0$. Thus f is differentiable at 0 and $f'(0) = 1$. However, f is not increasing on any interval containing 0. To see it, let $a < 0 < b$ so $0 \in (a, b)$. Using density of rational numbers pick a rational number q with $0 < q < \delta$. Then $q^2 > 0$ so $q^2 + q > q$. By density of irrational numbers, there is some irrational number z with $q < z < \min\{q^2 + q, b\}$. Then $q < z$ (and both are in (a, b)) but $f(q) = q^2 + q > z = f(z)$, so f is not increasing on (a, b).

However, not all is hopeless. The important ingredient we need is the following.

Theorem 25.4 (Mean Value Theorem)**.** *Assume f is continuous on the closed interval $[a, b]$ and differentiable at every point of (a, b). Then* *there exists a $c \in (a,b)$ such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. First we deal with the special case in which $f(a) = f(b)$.

If f is constant on $[a,b]$, then the derivative of f is zero at every point. By the Extreme Value Theorem, f attains a minimum and a maximum on $[a,b]$; say m and M, respectively. We have $m \le f(a) = f(b) \le M$. If m and M are both equal to $f(a)$ and $f(b)$, then f is constant on $[a,b]$, and we're done. Otherwise, either the minimum or maximum occurs at some c other than a and b, so in $c \in (a,b)$. Then the Corollary on Derivatives and Optimization says that $f'(c) = 0$.

Now, we no longer assume that $f(a) = f(b)$. Let $\ell(x) = \frac{f(b)-f(a)}{b-a}\, x$. Then for $g(x) = f(x) - \ell(x)$, we have

$$g(b)-g(a) = f(b)-f(a)-(\ell(b)-\ell(a)) = f(b)-f(a)-(\frac{f(b) - f(a)}{b - a})(b-a) = 0.$$

Since ℓ is continuous on $[a,b]$ and differentiable on (a,b), so is g, and the previous case implies that there exists a $c \in (a,b)$ with $g'(c) = 0$. But

$$0 = g'(c) = f'(c) - \ell'(c) = f'(c) - \frac{f(b) - f(a)}{b - a},$$

so $f'(c) = \frac{f(b)-f(a)}{b-a}$ as required. $\square$

Corollary 25.5. *Suppose I is an open interval (that is, $I = (a,b)$, (a,∞), $(-\infty,b)$, or (∞,∞)) and f is differentiable on all of I.*

(1) $f'(x) \ge 0$ for all $x \in I$ if and only if f is increasing on I.
(2) $f'(x) \le 0$ for all $x \in I$ if and only if f is decreasing on I.
(3) $f'(x) = 0$ for all $x \in I$ if and only if f is a constant on I.

Proof. We start with (1).

For the ($\Rightarrow$) direction, assume that $f'(x) \ge 0$ for all $x \in I$. To show that f is increasing, let $a, b \in I$ with $a < b$. Then f is differentiable on (a,b) since $(a,b) \subset I$ and f is continuous on $[a.b]$ since f is continuous on I and $[a.b] \subset I$. Thus, by the Mean Value Theorem, there is some $c \in (a,b)$ with $0 \le f'(c) = \frac{f(b)-f(a)}{b-a}$. Since $a < b$, $b - a > 0$ so we must have $f(b) - f(a) \ge 0$, so $f(a) \le f(b)$, as required.

For the ($\Leftarrow$) direction, we argue the contrapositive. Assume that $f'(r) < 0$ for all some $r \in I$. By Theorem ?, there is some $\delta > 0$ such that $f(x) < f(r)$ for all $x \in (x, x+\delta)$, so there is some $x \in I$ with $r < x$ and $f(x) > f(r)$. This implies that f is not increasing on I.

For (2), we can argue similarly or apply (1) to $-f(x)$.

For (3), we can argue similarly or observe that f is constant if and only if it is both increasing and decreasing on I, and that $f'(x) = 0$ on I if and only if $f'(x) \geq 0$ and $f'(x) \leq 0$ for all $x \in I$. $\square$

Discussion Questions.

(1) Using a Theorem, prove that $f(x) = x^3$ is increasing on $\mathbb{R}$.
(2) Using the definition and not theorems, prove that $f(x) = x^3$ is strictly increasing on $\mathbb{R}$.
(3) Prove or disprove: If $f'(r) = 0$, then there is some $a, b \in \mathbb{R}$ with $a < r < b$ such that f attains its maximum value on $[a, b]$ at $x = r$.
(4) Prove or disprove: Let f be differentiable on $\mathbb{R}$. If f is strictly increasing on $\mathbb{R}$, then $f'(x) > 0$ for all $x \in \mathbb{R}$.
(5) Prove or disprove: Let f be differentiable on $\mathbb{R}$. If $f'(x) > 0$ for all $x \in \mathbb{R}$, then f is strictly increasing on $\mathbb{R}$.

(1) We have $f'(x) = 3x^2 \geq 0$ for all $x \in \mathbb{R}$, so by the Corollary above, f is increasing on $\mathbb{R}$.
(2) Let $a < b$. Then $f(b) - f(a) = b^3 - a^3 = (b - a)(a^2 + ab + b^2)$. Then $a^2 + b^2 > 0$ unless $a = b = 0$ which is impossible, and $ab \geq 0$ unless $a < 0$ and $b > 0$, in which case the claim is clear, and if $a^2 + b^2 > 0$ and $ab \geq 0$, then $a^2 + ab + b^2 > 0$ and $b - a > 0$ implies $f(b) - f(a) > 0$, as required.
(3) False: Take $f(x) = x^3$ on $[-1, 1]$.
(4) False: Take $f(x) = x^3$ and $x = 0$.
(5) True: Let $f'(x) > 0$ for all $x \in \mathbb{R}$. Last $a < b$ be real numbers. Then f is differentiable on (a, b) and continuous on $[a, b]$ so the Mean Value Theorem applies. There is some $c \in (a, b)$ with $f'(c) = \frac{f(b) - f(a)}{b - a}$. Since $f'(c) > 0$ and $b - a > 0$, we must have $f(b) - f(a) > 0$ so $f(a) < f(b)$.

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