Name: Solutions

Instructions:

- You must show supporting work that determines how you arrived at your answer to receive full or partial credit.
- No textbook, notes, formula sheets, or calculators allowed.
- Unless indicated otherwise, your answers must be exact, not a numerical (decimal) approximation.
- If a problem statement includes physical quantities, then include units in your answer.
- If you want to write the solution to a problem on the blank page at the end, clearly indicate this by the prompt for the question.
- Take your time and read the questions carefully. Good luck!

Problem	Points	Score
1.	20	
2.	16	
3.	16	
4.	16	
5.	16	
6.	16	
Total	100	

1. Consider the autonomous differential equation

$$\frac{dy}{dx} = 4 - y^2.$$

(a) Is this equation *ordinary* or *partial*?

3 ordinary

(b) What is its order?

3 1

(c) Is it linear or nonlinear?

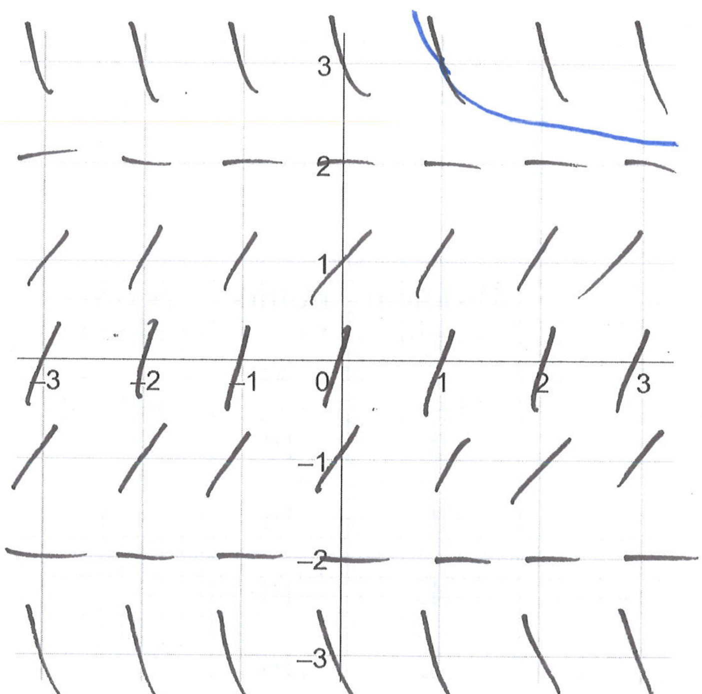
3 nonlinear

(d) Find all equilibrium solutions.

4 $4 - y^2 = 0 \rightarrow y = \pm 2$

(e) Sketch a slope field for this equation.

4



(f) For a solution of the equation with the initial condition $y(1) = 3$, what is $\lim_{x\to\infty} y(x)$?

3 2 (see blue curve)

5. A pond containing **1,000 L** of water is initially free of a pollutant. Water containing **0.1 g/L** of the pollutant flows into the pond at a rate of **250 L/h**, and water flows out of the pond at at a rate of **300 L/h**. Assume that the pollutant is uniformly distributed throughout the pond (i.e., it is well-mixed) at all times.

10

(a) Derive a differential equation for the amount of pollutant in the pond and write down the initial condition (but do not solve it).

P = amt pollutant in g

t = time in h

total water = $1000 - 300t + 250t = 1000 - 50t$

rate of change in pollutant

$= .1(250) - \left[\begin{matrix}\text{ratio pollutant}\\ \text{per L}\end{matrix}\right] \cdot 300$

$= 25 - \dfrac{P}{1000-50t} \cdot 300$

$$\Rightarrow \begin{cases} P' = 25 - \dfrac{300P}{1000-50t} \\ P(0) = 0. \end{cases}$$

6

(b) Find an explicit expression for the *integrating factor* of this differential equation.

(graded for consistency with the equation you found in (a) even if it was not the right equation)

$P' + \dfrac{300P}{1000-50t} = 25$

$P' + \dfrac{6P}{20-t} = 25$

$\mu = e^{\int \frac{6}{20-t}dt} = e^{-6\ln(20-t)} = e^{\ln((20-t)^{-6})} = \dfrac{1}{(20-t)^6}.$

4. Consider the differential equation $\frac{dy}{dx} = 3x\sqrt{y}$.

8

(a) Verify that the function $y = \frac{9}{16}x^4$ is a solution to this equation.

$$\left(\frac{9}{16}x^4\right)' \stackrel{?}{=} 3x\sqrt{\frac{9}{16}x^4}$$

$$\frac{9}{4}x^3 \stackrel{?}{=} 3x\left(\frac{3}{4}x^2\right)$$

yes

8

(b) Use Euler's method with step size 1 to approximate the solution at $x = 3$ of the initial value problem

$$\begin{cases} \frac{dy}{dx} = 3x\sqrt{y} \\ y(1) = 1 \end{cases}.$$

$x_0 = 1$ $\quad y_0 = 1$

$x_1 = 2$ $\quad y_1 = y_0 + (3x_0\sqrt{y_0})(1)$

$= 1 + 3 \cdot 1 \cdot 1 \cdot 1 = 4$

$x_2 = 3$ $\quad y_2 = y_1 + (3x_1\sqrt{y_1})(1)$

$= 4 + 3 \cdot 2 \cdot \sqrt{4} \cdot 1 = \boxed{16}$

3. Find a **particular** solution for

$$\begin{cases} \dfrac{dy}{dt} = \dfrac{\cos(t)}{y} \\ y(0) = 2 \end{cases}.$$

$$y\,dy = \cos(t)\,dt$$

$$\frac{y^2}{2} = \sin(t) + C$$

$$y^2 = 2\sin(t) + C \quad (\text{new } C)$$

$$y = \pm\sqrt{2\sin(t) + C}$$

$$2 = y(0) = \pm\sqrt{2\sin(0) + C} = \pm\sqrt{C}$$

$\Rightarrow C = 4$ and "+" not "−"

$$y = \sqrt{2\sin(t) + 4}.$$

2. Find the **general** solution to the following differential equation:

$$\frac{dy}{dx} = 2y + 1.$$

Separable:

$$\frac{dy}{2y+1} = dx$$

$$\int \frac{dy}{2y+1} = \int dx = x + C$$

$$= \frac{1}{2} \ln|2y+1| = \ln|2y+1|^{1/2}$$

$$|2y+1|^{1/2} = e^{x+C}$$

$$|2y+1| = (e^{x+C})^2 = e^{2x+2C} = Ce^{2x} \quad \text{(new } C\text{)}$$

$$2y+1 = \pm Ce^{2x} = Ce^{2x}$$

$$\underline{y = \frac{Ce^{2x} - 1}{2}}$$

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linear

$$y' - 2y = 1$$

$$\mu = e^{\int -2} = e^{-2x}$$

$$e^{-2x}y' - 2e^{-2x}y = e^{-2x}$$

$$= (e^{-2x}y)'$$

$$e^{-2x}y = \frac{-1}{2}e^{-2x} + C$$

$$y = \frac{-1}{2} + Ce^{2x}$$

6. Consider the differential equation $(t^2-4)y' = y^2 - 1$.

5

(a) Does there exist a unique differentiable function y on some interval around $t = 2$ such that

$$\begin{cases} (t^2-4)y' = y^2 - 1 \\ \text{and } \; y(2) = 5 \end{cases} ?$$

Why or why not?

~~No~~ because if $t=2$ & $y=5$

$(2^2-4)y' = 5^2 - 1$

$0y' = 24$ impossible

6

(b) Does there exist a unique differentiable function y on some interval around $t = 3$ such that

$$\begin{cases} (t^2-4)y' = y^2 - 1 \\ \text{and } \; y(3) = 1 \end{cases} ?$$

Why or why not?

PL: check $\frac{y^2-1}{t^2-4}$ & $\frac{\partial}{\partial y}\left(\frac{y^2-1}{t^2-4}\right) = \frac{2y}{t^2-4}$

cts near $(3,1)$

These are only ~~disc~~ discts when $t = \pm 2$.

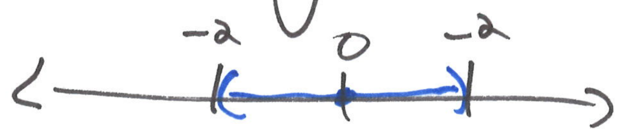
so YES

5

(c) What is the largest interval on which the Picard-Lindelöf Theorem guarantees the existence of a unique function y such that

$$\begin{cases} (t^2-4)y' = y^2 - 1 \\ \text{and } \; y(0) = 0 \end{cases} ?$$

From (b), PL says ok unless $t = \pm 2$



$(-2, 2)$

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