Schreyer Fall 2018

Review of Differentiation and Integration for Ordinary Differential Equations

In this course you will be expected to be able to differentiate and integrate quickly and accurately. Many students take this course after having taken their previous course many years ago, at another institution where certain topics may have been omitted, or just feel uncomfortable with particular techniques. Because understanding this material is so important to being successful in this course, we have put together this brief review packet.

In this packet you will find sample questions and a brief discussion of each topic. If you find the material in this pamphlet is not sufficient for you, it may be necessary for you to use additional resources, such as a calculus textbook or online materials. Because this is considered prerequisite material, it is ultimately your responsibility to learn it. The topics to be covered include Differentiation and Integration.

1 Differentiation

Exercises:

1. Find the derivative of $y = x^3 \sin(x)$.
2. Find the derivative of $y = \frac{\ln(x)}{\cos(x)}$.
3. Find the derivative of $y = \ln(\sin(e^{2x}))$.

Discussion:
It is expected that you know, without looking at a table, the following differentiation rules:

$$\frac{d}{dx}\left[(kx)^n\right] = kn(kx)^{n-1} \tag{1}$$

$$\frac{d}{dx}\left[e^{kx}\right] = ke^{kx} \tag{2}$$

$$\frac{d}{dx}\left[\ln(kx)\right] = \frac{1}{x} \tag{3}$$

$$\frac{d}{dx}\left[\sin(kx)\right] = k\cos kx \tag{4}$$

$$\frac{d}{dx}\left[\cos(kx)\right] = -k\sin x \tag{5}$$

$$\frac{d}{dx}\left[uv\right] = u'v + uv' \tag{6}$$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2} \tag{7}$$

$$\frac{d}{dx}\left[u(v(x))\right] = u'(v)v'(x). \tag{8}$$

We put in the constant k into (1) - (5) because a very common mistake to make is something like: $\frac{d}{dx}e^{2x} = \frac{e^{2x}}{2}$ (when the correct answer is $2e^{2x}$). Equation (6) is known as the product rule, Equation (7) is known as the quotient rule, and Equation (8) is known as the chain rule. From these, you can derive the derivative of many other functions, such as the tangent:

$$\begin{aligned}
\tan(x) &= \frac{\sin(x)}{\cos(x)} \\
\frac{d}{dx}\left[\tan(x)\right] &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{(\cos(x))^2} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
&= \sec^2(x)
\end{aligned}$$

where we have used the quotient rule and simplified.

The chain rule is applied when there is a *function of a function*, i.e. $f(g(x))$. The idea is to take the derivative of the outside function first, leaving its argument alone. Then multiply that by the derivative of the next outermost function, leaving it's argument alone. The process is repeated until there is nothing left of which to take the derivative. So for example, to take the derivative of $\sin^2(5x)$, we need to first determine the outside function. If we re-write it as $(\sin(5x))^2$ it is quickly determined that the outside function is "something squared", where "something" in this case is $\sin(5x)$. The derivative of "something squared" is "2 times that something times the derivative of that something". Thus we have

$$\begin{aligned}
\tfrac{d}{dx}\left[\sin^2(5x)\right] &= \frac{d}{dx}\left[(\sin(5x))^2\right] \\
&= 2\sin(5x)\frac{d}{dx}\left[\sin(5x)\right] \\
&= 2\sin(5x)\cos(5x)\frac{d}{dx}\left[5x\right] \\
&= 2\sin(5x)\cos(5x)5 \\
&= 10\sin(5x)\cos(5x)
\end{aligned}$$

Solution to Exercises:

1. For this problem, we need the product rule, (6), since two functions are being multiplied. In this case, $u(x) = x^3$ and $v(x) = \sin(x)$. Thus,

$$\frac{d}{dx}\left[x^3\sin(x)\right] = 3x^2\sin(x) + x^3[\cos(x)]$$

$$= 3x^2 \sin(x) + x^3 \cos(x).$$

2. This is clearly a quotient of functions, so that the quotient rule applies, (7). We have $u(x) = \ln(x)$ and $v(x) = \cos(x)$, which implies:

$$\begin{aligned}
\frac{d}{dx}\left[\frac{\ln(x)}{\cos(x)}\right] &= \frac{\frac{1}{x}\cos(x) - \ln(x)[-\sin(x)]}{(\cos(x))^2} \\
&= \frac{\frac{1}{x}\cos(x) + \ln(x)\sin(x)}{\cos^2(x)} \\
&= \frac{\cos(x) + x\ln(x)\sin(x)}{x\cos^2(x)}
\end{aligned}$$

3. This is a case of a function of a function of a function of a function, $f(g(h(i(x))))$. We apply the chain rule, always working from the outside function in. In this case the (very) outside function is $f() = \ln$ of "something"; the next most outside function is, $g() = \sin$ of "something", the next most outside function is, $h() = e$ to the "something", and the inside most function is $i(x) = 2x$. Applying the chain rule (8) we have

$$\begin{aligned}
\frac{d}{dx}\left[\ln(\sin(e^{2x}))\right] &= \frac{1}{\sin(e^{2x})}\frac{d}{dx}\left[\sin(e^{2x})\right] \\
&= \frac{1}{\sin(e^{2x})}\cos(e^{2x})\frac{d}{dx}\left[e^{2x}\right] \\
&= \frac{1}{\sin(e^{2x})}\cos(e^{2x})e^{2x}\frac{d}{dx}\left[2x\right] \\
&= \frac{1}{\sin(e^{2x})}\cos(e^{2x})e^{2x}2 \\
&= \frac{2e^{2x}\cos(e^{2x})}{\sin(e^{2x})} \\
&= 2e^{2x}\cot(e^{2x})
\end{aligned}$$

2 Integration

Solving differential equations requires integration - there's just no getting around it. What follows is a brief review. If you need supplemental material, please see your calculus text.

Exercises:

1. Evaluate $\displaystyle\int x\sqrt{x^2+1}\, dx$.

2. Evaluate $\displaystyle\int \frac{\sin(x)}{\cos(x)}\, dx$.

3. Evaluate $\int xe^{3x}\, dx$

4. Given $\int u^n \ln(u)\, du = \dfrac{u^{n+1}\ln(u)}{n+1} - \dfrac{u^{n+1}}{(n+1)^2} + C$, evaluate $\int x^2 \ln(2x)\, dx$.

The discussion section is rather long, and the solution to these exercises are given at the end of this section.

Discussion:

2.1 Basic Integration Formulas

Not only is it important to be familiar with various integration techniques, but it is also important that we be quick and efficient when evaluating integrals so that we can concentrate on the concepts as oppose to the mechanics of integration.

Examples

1. We can evaluate the indefinite integral $\int e^{2x}dx$ by doing a *u-substitution*. However, we can become more efficient at evaluating integrals of this type by obtaining a general formula. Let $f(x) = e^{ax}$, where a is equal to a constant. We would like to obtain a general formula for $\int e^{ax}dx$. We can accomplish this by doing a u-substitution. Let $u = ax$, then $du = adx \Rightarrow \frac{du}{a} = dx$. Substitution yields

   $$\int e^{ax}dx = \tfrac{1}{a}\int e^u du = \tfrac{1}{a}e^u + C = \tfrac{1}{a}e^{ax} + C$$

   Now we have a general formula that we can use again and again without going to the trouble of doing the u-substitution each time. For example,

   (a) $y(x) = e^{2x}$

   $$\int e^{2x}dx = \frac{1}{2}e^{2x} + C$$

   (b) $y(x) = e^{\pi x}$

   $$\int e^{\pi x}dx = \frac{1}{\pi}e^{\pi x} + C$$

2. Let us follow the procedure in *Example 1* to find the general formula for integrals of the form $\int \cos(ax)\, dx$, where a is equal to a constant. Let $u = ax$, then $du = adx \Rightarrow \frac{du}{a} = dx$. Substitution yields

$$\int \cos(ax)\ dx = \tfrac{1}{a}\int \cos(u)\ du = \tfrac{1}{a}\sin(u) + C = \tfrac{1}{a}\sin(ax) + C.$$

Here are some examples of using this general formula.

(a) $y(x) = \cos(4x)$

$$\int \cos(4x)\ dx = \frac{1}{4}\sin(4x) + C$$

(b) $y(x) = \cos\frac{1}{2\pi}x$

$$\int \cos\left(\frac{1}{2\pi}x\right) dx = 2\pi \sin 2\pi x + C$$

Exercises

Follow the examples above to obtain a general formula for the integral given, then use it to evaluate parts (a) and (b). As above, a is equal to a constant.

1. $\displaystyle\int \sin(ax)\ dx$

   (a) $\displaystyle\int \sin(16x)\ dx$

   (b) $\displaystyle\int \sin\left(\frac{1}{2}x\right)\ dx$

2. $\displaystyle\int \ln(ax)\ dx$

   (a) $\displaystyle\int \ln(\pi x)\ dx$

   (b) $\displaystyle\int \ln(\frac{1}{\pi}x)\ dx$

3. $\displaystyle\int \tan(ax)\ dx$

   (a) $\displaystyle\int \tan(3x)\ dx$

   (b) $\displaystyle\int \tan\left(\frac{1}{3}x\right)\ dx$

4. $\displaystyle\int \sec(ax)\ dx$

   (a) $\int \sec(2.78x)\ dx$

   (b) $\int \sec(1618x)\ dx$

5. $\int \arctan(ax)\ dx$

(a) $\int \arctan(\pi x)\ dx$

(b) $\int \arctan\left(\frac{1}{\pi}x\right)\ dx$

2.2 u-substitution

In general, u-substitutions are not as straight forward as the ones in the previous section. When doing a u-substitution you want to look for the part of the integral whose derivative is elsewhere in the integral (up to a constant). Formally, if we have an integral of the form

$$\int f(g(x))g^{'}(x)dx,$$

we let $u = g(x)$, then $du = g^{'}(x)dx$, substitution yields

$$\int f(g(x))g'(x)dx = \int f(u)du$$

Essentially, we have transformed the space in which we are evaluating the integral. We evaluate the integral in this new space and then substitute u back in to obtain a solution in the original space. Similar techniques are often employed to solve differential equations.

Examples

1. $\int x^5 e^{x^6} dx$
   First, let $u = x^6$, then $du = 6x^5 \Rightarrow \frac{du}{6} = x^5$. Substitution yields

$$\begin{aligned} \int x^5 e^{x^6} dx &= \frac{1}{6}\int e^u du \\ &= \frac{1}{6}e^u + C \\ &= \frac{1}{6}e^{x^6} + C. \end{aligned}$$

2. $\int (x^2+1)^2(2x)\ dx$
   First, let $u = x^2 + 1$, then $du = 2x\ dx$. Substitution yields

$$\begin{aligned} \int (x^2+1)^2(2x)dx &= \int u^2\ du \\ &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(x^2+1)^3 + C. \end{aligned}$$

Exercises

Use u-substitution to evaluate the following indefinite integrals.

1. $\int \sin^2(3x)\cos(3x)\ dx$
2. $\int \frac{1}{\theta^2}\cos\left(\frac{1}{\theta}\right)\ d\theta$
3. $\int \frac{\sin x}{\cos^2 x}\ dx$
4. $\int e^x(e^x+1)^2\ dx$
5. $\int \tan^4 x \sec^2 x\ dx$

2.3 Integration by Parts

Integration by parts is applicable to a plethora of functions which we may need to integrate. Formally, if u and v are functions of x and have continuous derivatives, then

$$\int udv = uv - \int vdu$$

Choosing which part is equal to u may be facilitated by remembering the acronym: LIATE, which stands for: Logarithm, Inverse trig, Algebraic, Trigonometric, Exponential. This means that whichever of these expressions appears first in the acronym, that is the expression you should let u be. So if you want to evaluate $\int(x^2+5x-2)e^{5x}\ dx$, we see we have an algebraic expression, x^2+5x-2, times an exponential function, e^{5x}. By this acronym, since A appears before E, we set $u = x^2+5x-2$.

Examples

1. Evaluate $\int xe^x\ dx$

   First, let $u = x \Rightarrow du = dx$ and let $dv = e^x dx \Rightarrow v = e^x$. Using the integration by parts formula we obtain

$$\begin{aligned}\int xe^x\ dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C.\end{aligned}$$

2. Evaluate $\int \arcsin x dx$

   First, let $u = \arcsin x \Rightarrow du = \frac{1}{\sqrt{1-x^2}}dx$ and let $dv = dx \Rightarrow v = x$.

$$\int \arcsin x dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx$$

$$
\begin{aligned}
&= x\arcsin x + \frac{1}{2}\int w^{-\frac{1}{2}}dw \\
&= x\arcsin x + w^{\frac{1}{2}} + C \\
&= x\arcsin x + (1-x^2)^{\frac{1}{2}} + C \\
&= x\arcsin x + \sqrt{1-x^2} + C.
\end{aligned}
$$

Where the second equality comes from doing a $u-$*substitution* (w in this case) where $w = 1 - x^2$

3. Sometimes it is necessary to do integration by parts more than once. For example, $\int x^2 e^x dx$.
   First, let $u = x^2 \Rightarrow du = 2xdx$ and let $dv = e^x dx \Rightarrow v = e^x$. Substitution yields

$$
\begin{aligned}
\int x^2 e^x dx &= x^2 e^x - 2\int x e^x dx \\
&= x^2 e^x - 2(xe^x - e^x) + C \\
&= x^2 e^x - 2xe^x - 2e^x + C.
\end{aligned}
$$

   where the second equality comes from our previous calculation in *Example 1.*

4. Here is another example where integration by parts will be used repeatedly to evaluate an integral. Evaluate $y(x) = e^x \cos x$.
   First, let $u = e^x \Rightarrow du = e^x dx$ and let $dv = \cos x\ dx \Rightarrow v = \sin x$. Substitution yields

$$
\int e^x \sin x\ dx = e^x \sin x - \int e^x \sin x dx.
$$

   Using integration by parts again, let $u = e^x \Rightarrow du = e^x dx$ and let $dv = \sin x\ dx \Rightarrow v = -\cos x$. Substitution yields

$$
\begin{aligned}
\int e^x \cos x\ dx &= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x\ dx\right] \\
&= e^x \sin x + e^x \cos x - \int e^x \cos x\ dx \\
2\int e^x \cos x\ dx &= e^x \sin x + e^x \cos x \\
\int e^x \cos x\ dx &= \frac{1}{2}(e^x \sin x + e^x \cos x) + C.
\end{aligned}
$$

Exercises

Use integration by parts to evaluate the following indefinite integrals.

1. $\int t \ln(t+1)\ dt$
2. $\int \frac{(\ln x)^2}{x}\ dx$
3. $\int \arccos x\ dx$

4. $\int e^{2x} \sin x \, dx$

5. $\int e^{x} \sin x \, dx$

2.4 Integration using Partial Fraction Decomposition

Partial fraction decomposition is applicable to *rational* functions which we may need to integrate. A rational function is one which is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}.$$

We consider two cases: The degree of $P(x) <$ degree of $Q(x)$ and the degree of $P(x) \geq$ degree of $Q(x)$. If the degree of $P(x) \geq Q(x)$, then we use long division to write the rational polynomial as a polynomial plus a rational polynomial where the degree of the numerator is less than the degree of the denominator.

Long Division

Examples

1. $\dfrac{x^3 - x - 3\sqrt{2}}{x^2 - 2}$
   We note that the degree of the numerator, 3, is larger than the degree of the denominator, 2. Using long division:

$$(x^2 - 2)\overline{\big)x^3 - x - 3\sqrt{2}}$$

   Think: since x^2 goes into x^3 x times, and we have:

$$\begin{array}{rl} & \quad x \\ x^2 - 2 & \overline{\big) x^3 - x - 3\sqrt{2}} \\ & \underline{-(x^3 - 2x)} \\ & \qquad x - 3\sqrt{2} \end{array}$$

   Since the degree of $x - 3\sqrt{2}$ is less than the degree of $x^2 - 2$ we stop and we have:

$$\frac{x^3 - x - 3\sqrt{2}}{x^2 - 2} = x + \frac{x - 3\sqrt{2}}{x^2 - 2}$$

2. $\dfrac{x^5 - 3}{x^3 - x^2}$
   We note that the degree of the numerator, 5, is larger than the degree of the denominator, 3. Using long division:

$$(x^3 - x^2)\overline{\big)x^5 - x^3}$$

Think: x^3 goes into x^5 x^2 times, and we have

$$\begin{array}{rl} & \quad x^2 \\ x^3 - x^2 & \sqrt{x^5 - 3} \\ & \underline{-(x^5 - x^4)} \\ & \qquad x^4 - 3 \end{array}$$

Now x^3 goes into x^4 x times and we continue:

$$\begin{array}{rl} & \quad x^2 + x + 1 \\ x^3 - x^2 & \sqrt{x^5 - 3} \\ & \underline{-(x^5 - x^4)} \\ & \qquad x^4 - 3 \\ & \qquad \underline{-(x^4 - x^3)} \\ & \qquad\quad x^3 - 3 \\ & \qquad\quad \underline{-(x^3 - x^2)} \\ & \qquad\qquad x^2 - 3 \end{array}$$

and so we have

$$\frac{x^5 - 3}{x^3 - x^2} = x^2 + x + 1 + \frac{x^2 - 3}{x^3 - x^2}.$$

We can check our answer by doing a quick polynomial addition where we put everything under the same denominator:

$$\frac{(x^2 + x + 1)(x^3 - x^2)}{x^3 - x^2} + \frac{x^2 - 3}{x^3 - x^2} = \frac{x^5 - 3}{x^3 - x^2}.$$

Factoring Polynomials

Since integrating polynomials is (hopefully) easy, we now concentrate on integrating rational functions where the degree of the numerator is strictly less than the degree of the denominator. We first note that

All polynomials with real coefficients can be factored into linear and irreducible quadratic factors:

$$Q(x) = (\alpha_1 x - \beta_1)^{n_1} (\alpha_2 x - \beta_2)^{n_2} \ldots (a_1 x^2 + b_1 x + c_1)^{m_1} (a_2 x^2 + b_2 x + c_2)^{m_2} \ldots$$

An irreducible quadratic factor is one that has imaginary roots, i.e. from the quadratic formula, $b^2 - 4ac < 0$. So for example

$$x^4 - 5x^2 - 5 = (x^2 - 1)(x^2 + 4) = (x - 1)(x + 1)(x^2 + 4)$$

Here the linear factors are $(x-1)$ and $(x+1)$ and the irreducible quadratic factor is (x^2+4) since the roots of (x^2+4) are $\pm 2i$ (complex). Note that (x^2-1) is *not* irreducible, because it has real roots (± 1).

Another example:

$$2x^4 + 2x^3 + 3x^2 = x^2(5x^2 + 2x + 3).$$

Here we have a repeated linear root: $x = 0$ appears twice (think $(x-0)^2$), and an irreducible quadratic: for $5x^2 + 2x + 3$, $b^2 - 4ac = 5^2 - 4(5)(3) = -35 < 0$, so this quadratic has only complex roots.

Partial Fraction Decomposition

We use the following examples to guide our selection of how to decompose a rational function where the degree of the numerator is strictly less than the degree of the denominator. Note that the denominators have already been factored into linear and irreducible quadratic terms.
Examples

1. $\dfrac{1}{(x-1)(x-2)(x-3)} = \dfrac{A}{x-1} + \dfrac{B}{x-2} + \dfrac{C}{x-3}$

2. $\dfrac{1}{(x-1)^3(x-2)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3} + \dfrac{D}{x-2}$

3. $\dfrac{1}{x^2(x^2+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+1}$

4. $\dfrac{1}{(x-1)^2(x^2+1)^2} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1} + \dfrac{Ex+F}{(x^2+1)^2}$

The goal is to determine the values of the parameters A, B, ... F and then we can integrate each term using techniques we already know. Note that in each case, the number of unknown parameters is equal to the degree of the original polynomial in the denominator. In example 1, on the left side, the denominator is a polynomial of degree 3, and there are 3 parameters, A, B, and C. In example 4, the, the degree of the polynomial in the denominator is 6, and there are 6 parameters, etc. This is a sanity check, to be sure we have not made a blatant error in setting up the partial fraction decomposition.

There are two methods that are used to determine the unknown parameters, and we will do two examples illustrating each method. In some problems, the first method is easiest to use, and in others the second. Only experience will help you determine which method to use.

Examples

1. Find the partial fraction decomposition of

$$\frac{1}{(x-1)(x-2)(x-3)}.$$

We first note that the degree of the numerator (0) is less than the degree of the denominator (3). The denominator is already factored, and from the discussion above we know that the form of the partial fraction decomposition is

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} \tag{9}$$

First get the denominators of each term the same by multiplying each term on the right-hand side by special forms of 1:

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}\frac{(x-2)(x-3)}{(x-2)(x-3)} + \frac{B}{x-2}\frac{(x-1)(x-3)}{(x-1)(x-3)} + \frac{C}{x-3}\frac{(x-1)(x-2)}{(x-1)(x-2)}. \tag{10}$$

Because the denominators are the same, in order to satisfy the above equation, we now only need to be sure the numerators are equal. Examining the numerators only we have

$$1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2). \tag{11}$$

The above equation must hold for all x. Since it holds for all x, it must hold for particular values of x, and we choose these values of x to make it easy to solve for A, B, and C.

Consider $x = 1$. Substituting 1 in for x into equation (11) we get:

$$1 = A(-1)(-2) + B(0)(-2) + C(0)(-1) = 2A \quad \Rightarrow \quad A = \frac{1}{2}$$

Likewise, let us consider $x = 2$ and $x = 3$. Substituting these values into equation (11) gives us respectively,

$$\begin{aligned} 1 &= A(0)(-1) + B(1)(-1) + C(1)(0) = -B \quad \Rightarrow \quad B = -1 \\ 1 &= A(1)(0) + B(2)(0) + C(2)(1) = 2C \quad \Rightarrow \quad C = \frac{1}{2}. \end{aligned}$$

Thus we have (going back to equation (9)):

$$\frac{1}{(x-1)(x-2)(x-3)} = \frac{1/2}{x-1} + \frac{-1}{x-2} + \frac{1/2}{x-3}$$

where we see that the right-hand side is much easier to integrate than the left side. We can check our answer by adding the fractions on the right-hand side (requiring us to get a common denominator) and checking to be sure we get the left-hand side.

Note that in general, if we want to solve for 2 unknowns, we need 2 x values; if we want to solve for 3 unknowns, we need 3 x values; etc. This is because if we want to solve for n unknowns we need n equations that are not redundant.

2. Find the partial fraction decomposition of

$$\frac{3x^3 + x - 2}{x^2(x^2+1)}$$

We first note that the degree of the numerator (3) is less than the degree of the denominator (4). The denominator is already factored, and from the discussion above we know that the form of the partial fraction decomposition is

$$\frac{3x^3 + x - 2}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2+1} \tag{12}$$

First get the denominators of each term the same by multiplying each term on the right-hand side by special forms of 1:

$$\frac{3x^3 + x - 2}{x^2(x^2+1)} = \frac{A}{x}\,\frac{x(x^2+1)}{x(x^2+1)} + \frac{B}{x^2}\,\frac{(x^2+1)}{(x^2+1)} + \frac{Cx + D}{x^2+1}\,\frac{x^2}{x^2}. \tag{13}$$

Because the denominators are the same, in order to satisfy the above equation, we now only need to be sure the numerators are equal. Examining the numerators only we have

$$3x^3 + x - 2 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2 \tag{14}$$
$$= Ax^3 + Ax + Bx^2 + B + Cx^3 + Dx^2 \tag{15}$$

The above equation says that we must find the values of A, B, C, and D so that the two polynomials (the one on the left side and the one on the right) are equal. *Two polynomials are equal if and only if their coefficients are equal.* So we look at the coefficients of x^3, x^2, x, and x^0 (constants), to give us 4 equations for the 4 unknowns:

$$\begin{aligned} x^3:&\quad 3 = A + C \\ x^2:&\quad 0 = B + D \\ x^1:&\quad 1 = A \\ x^0:&\quad -2 = B \end{aligned}$$

It turns out that these 4 equations are easy to solve for our unknowns, but in general one can use the technique of elimination to solve for the 4 unknowns. In this case, the third and fourth equations tell us that $A = 1$ and $B = -2$. Using these values in the first two equations gives $C = 3 - 1 = 2$ and $D = -B = 2$. Using these values, we can now write equation (12) as:

$$\frac{3x^3 + x - 2}{x^2(x^2+1)} = \frac{1}{x} + \frac{-2}{x^2} + \frac{2x + 2}{x^2+1}$$

To integrate the last term we rewrite the fraction as:

$$\frac{2x+2}{x^2+1} = \frac{2x}{x^2+1} + \frac{2}{x^2+1}.$$

The first of these terms can be integrated using u-substitution, and the second is of the form $\frac{1}{u^2+a^2}$ which has an integral of $\frac{1}{a}\tan^{-1}\frac{u}{a} + C$.

Problems

1. Find the partial fraction decomposition of

$$\frac{x^3+1}{(x^2+1)^2}$$

   Solve for the unknown coefficients, but do *not* integrate.

2. Evaluate

$$\int \frac{x-3}{x^2-3x+3}\,dx$$

3. Integrate

$$\frac{x^2+2}{(x^2+x+1)(x-1)}$$

   (Hint: to integrate one of the terms, consider completing the square)

4. Integrate

$$\frac{x^3-x-3\sqrt{2}}{x^2-2}$$

5. What happens when you try to solve for A, B, and C for the partial fraction decomposition:

$$\frac{1}{(x^2-1)(x-1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2-1}?$$

   Try to solve for A, B, and C, and show all work. What is wrong with this formulation?