1. Consider the equation

$$y'' - 4y' + 4y = 0.$$

(a) Use the auxiliary equation to determine two solutions.

8

$r^2 - 4r + 4 = 0$

$(r-2)^2 = 0$

$e^{2t}, \quad te^{2t}$

(b) **Using the Wronskian**, verify that two solutions you found are linearly independent.

7

$$\begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} \end{vmatrix} = e^{4t} + 2te^{4t} - 2te^{4t}$$

$= e^{4t} \neq 0$

yes

(c) Find the solution to this equation that also satisfies the initial conditions $y(0) = 0$ and $y'(0) = 1$.

7

$y = C_1 e^{2t} + C_2 te^{2t} \qquad y' = 2C_1 e^{2t} + C_2 e^{2t} + 2C_2 te^{2t}$

$0 = C_1 e^{0} + C_2 0 e^0 \Rightarrow C_1 = 0$

$1 = C_2 e^0 \Rightarrow C_2 = 1$

$y = te^{2t}$

2. Find a particular solution to the differential equation

$$y'' - y = e^t$$

given that e^t and e^{-t} are two solutions to the corresponding homogeneous equation.

$y_p = Ate^t$

$y_p' = A(e^t + te^t)$

$y_p'' = A(e^t + e^t + te^t)$

$= Ae^t(t+2)$

$y_p'' - y_p = 2Ae^t \quad A = 1/2$

$y_p = \frac{1}{2}te^t$

3. (a) The displacement from resting position in a particular mass-spring system with friction/damping is given by the differential equation

$$y'' + 3y' + 5y = 0.$$

8

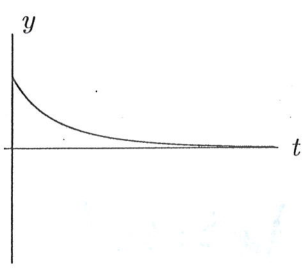
Is this system **overdamped**, **critically damped**, or **underdamped**? Explain your answer.

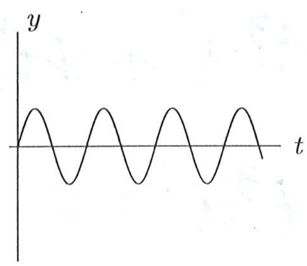
$3^2 - 4 \cdot 1 \cdot 5 = 9 - 20 < 0$

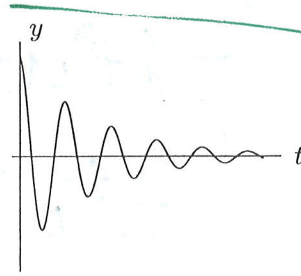
underdamped

(b) Circle the graph that could be a solution for the equation in part (a). (No explanation necessary.)

5





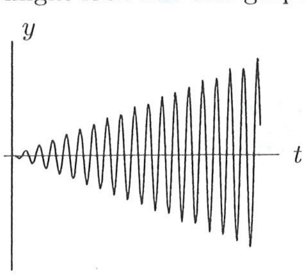


(c) Give a simple function $f(t)$ such that a solution of the equation

$$y'' + 5y = f(t)$$

might look like this graph:

3



$\sin(\sqrt{5}t)$ (resonance, or b/c we get a trial solution like $At\sin\sqrt{5}t + Bt\cos\sqrt{5}t$)

4. Consider the differential equation
$$t^2y'' - 3ty' + 4y = 0.$$
The function $y_1(t) = t^2$ is a solution to this equation. Use the method of **reduction of order** to find another solution $y_2(t)$ that is linearly independent with $y_1(t)$.

$y_2 = t^2 u$

$y' = 2tu + t^2u'$

$y'' = 2u + 2tu' + 2tu' + t^2u''$

$= 2u + 4tu' + t^2u''$

$t^2y'' - 3ty' + 4y$

$\shortparallel$

$\cancel{2ut^2} + 4t^3u' + t^4u''$

$- \cancel{6}t^2u - 3t^3u' + \cancel{4}t^2u$

$t^4u'' + t^3u' = 0$ $\qquad v = u'$

$tv' + \cancel{t}v = 0$

$t\dfrac{dv}{dt} = -v \qquad \dfrac{dv}{v} = \dfrac{-dt}{t}$

$\ln v = -\ln t$

$V = \cancel{e}\, e^{\cancel{\ln} 1/t} = 1/t$

$u = \ln t$

$y_2 = t^2 \ln t.$

5. (a) Compute the Laplace transform $\mathcal{L}\{f(t)\}$ of the function $f(t) = 2\cos(3t) - 3e^{7t}$.

5 $2\frac{s}{s^2+9} - 3\frac{1}{s-7}$

(b) Given the initial value problem below, apply the Laplace transform to get an algebraic equation for $Y = \mathcal{L}\{y\}$ (the Laplace transform of the solution). **You do not need to solve or simplify the equation that you get.**

$$\begin{cases} y' - 2y = 5\sin(t) \\ y'(0) = 4 \end{cases}.$$

5 $sY - 4 - 2Y = \frac{5}{s^2+1}$

(c) Compute the inverse Laplace transform $\mathcal{L}^{-1}\{F(s)\}$ of the function $F(s) = \dfrac{1}{(s-2)(s+3)}$.

$\frac{1}{(s-2)(s+3)} = \frac{A}{s-2} + \frac{B}{s+3}$

$1 = (s+3)A + (s-2)B$

6

$s=2 \rightsquigarrow 5A = 1$

$s=3 \rightsquigarrow -5B = 1$

$\frac{1/5}{s-2} + \frac{-1/5}{s+3}$

$\frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}$

6. (a) Suppose that $f(t)$ is a solution to the differential equation

$$(\clubsuit) \qquad y^{(3)} - 3\sin(t)y' = e^{t^2}$$

and that $g(t)$ and $h(t)$ are solutions to the differential equation

$$(\diamondsuit) \qquad y^{(3)} - 3\sin(t)y' = 0.$$

6

Circle all of the following statements that are **true**.

+2 right
-1 wrong

- $3f(t)$ is a solution to $(\clubsuit)$
- $f(t)^2$ is a solution to $(\clubsuit)$
- $6h(t) + f(t)$ is a solution to $(\clubsuit)$
- $3g(t)$ is a solution to $(\diamondsuit)$
- $2g(t) + 7h(t)$ is a solution to $(\diamondsuit)$
- $tg(t) + 7h(t)$ is a solution to $(\diamondsuit)$

(b) Consider the initial value problem

$$\begin{cases} y^{(3)} - 3\sin(t)y' = e^{t^2} \\ y(1) = a, \; y'(1) = b, \; y''(1) = c \end{cases}$$

where a, b, c are some constants. Circle all of the following statements that are **true**.

6

- 3 No matter what a, b, c are, there *is* a solution to this IVP.
- For some choices of a, b, c, there *is* a solution, and for some choices, there is *no* solution.
- No matter what a, b, c are, there is *no* solution to this IVP.
- 3 No matter what a, b, c are, there is *at most one* solution to this IVP.
- For some choices of a, b, c, there is *at most one* solution, and for some choices, there is *more than one* solution.
- No matter what a, b, c are, there is *more than one* solution to this IVP.

(c) Consider the initial value problem

$$\begin{cases} y'' - 3\sin(t)y' = e^{t^2} \\ y(1) = a, \; y'(1) = b, \; y''(1) = c \end{cases}$$

where a, b, c are some constants. *Note that the differential equation in this IVP is different from the one above.* Circle all of the following statements that are **true**.

4

- No matter what a, b, c are, there *is* a solution to this IVP.
- 2 For some choices of a, b, c, there *is* a solution, and for some choices, there is *no* solution.
- No matter what a, b, c are, there is *no* solution to this IVP.
- No matter what a, b, c are, there is *at most one* solution to this IVP.
- 2 For some choices of a, b, c, there is *at most one* solution, and for some choices, there is *more than one* solution.
- No matter what a, b, c are, there is *more than one* solution to this IVP.