MIDTERM EXAM

For this exam, you may use the lecture notes, your own notes, and the homework assignments, but you should not use other sources, including your classmates. However, if you plan to take a written algebra comp, I recommend that you first attempt the problems without notes, and then complete them with notes later. Turn in solutions to *five* of the problems below.

(1) (a) Let P be a poset. Show that every morphism in $\mathbf{PO}(P)$ is both monic and epic.
 (b) Give an example of a poset P, a morphism $\alpha \in \mathbf{PO}(P)$, and a functor $F : \mathbf{PO}(P) \to \mathscr{C}$ to some category $\mathscr{C}$ such that $F(\alpha)$ is neither monic nor epic.

(2) Let K be a field, and $R = M_n(K)$ be the ring of $n \times n$ matrices over K. Let M be the right R-module of $1 \times n$ row vectors and N be the left R-module of $n \times 1$ column vectors. Observe that M and N are also K-vectorspaces.
 (a) Show that $M \otimes_K N \cong R$ (as abelian groups).
 (b) Show that $M \otimes_R N \cong K$ (as abelian groups).

(3) Let K be a field. Show that any additive functor $F : K - \mathbf{Vect} \to K - \mathbf{Vect}$ is exact.

(4) Let $\mathbf{Set}^\infty$ be the full subcategory of $\mathbf{Set}$ consisting of all infinite sets. Let $F : \mathbf{Set}^\infty \to \mathbf{Set}^\infty$ be the functor that on objects is given by the rule $F(S) = S \times S$, and on morphisms is given by $F(f) = (f, f)$. Show that there is no natural isomorphism $\eta : F \Rightarrow 1_{\mathbf{Set}^\infty}$.

(5) Give examples of $\mathbb{Z}$-modules with the following sets of properties, and justify.
 (a) Injective, but not flat.
 (b) Flat and injective, but not projective.

(6) Let $\phi : R \to S$ be a ring homomorphism. Let N be an R-module, and M be an S-module. Let $F : S - \mathbf{Mod} \to R - \mathbf{Mod}$ be restriction of scalars along ϕ, and $G : R - \mathbf{Mod} \to S - \mathbf{Mod}$ be the functor $\mathrm{Hom}_R(S, -)$. Show that $\mathrm{Hom}_R(F(M), N) \cong \mathrm{Hom}_S(M, G(N))$.

(7) Let R be a commutative ring, and M, N be R-modules.
 (a) Show that if M and N are projective, then $M \otimes_R N$ is projective.
 (b) Show that if M and N are flat, then $M \otimes_R N$ is flat.