ASSIGNMENT #5

Let $F : R-\mathbf{Mod} \to S-\mathbf{Mod}$ be a covariant functor. Then F extends to a covariant functor from $R-\mathbf{Comp} \to S-\mathbf{Comp}$ by mapping

$$F\left(\cdots \to M_{i+1} \xrightarrow{d_{i+1}} M_i \xrightarrow{d_i} M_{i-1} \to \cdots \right) = \cdots \to F(M_{i+1}) \xrightarrow{F(d_{i+1})} F(M_i) \xrightarrow{F(d_i)} F(M_{i-1}) \to \cdots$$

and for a chain map $\{f_i\}$, $F(\{f_i\}) = \{F(f_i)\}$.

(1) Let $F : R-\mathbf{Mod} \to S-\mathbf{Mod}$ be a covariant additive functor. Show that[1] if $F : R-\mathbf{Mod} \to S-\mathbf{Mod}$ is exact, then for every chain complex $C_\bullet$, there are isomorphisms $H_i(F(C_\bullet)) \cong F(H_i(C_\bullet))$.

(2) Show that $F : R-\mathbf{Mod} \to S-\mathbf{Mod}$ be a covariant additive functor and $f, g : M_\bullet \to N_\bullet$ are homotopic maps, then $F(f)$ and $F(g)$ are also homotopic. Conclude that if the identity map of $M_\bullet$ is nullhomotopic, then $F(M_\bullet)$ is exact for any covariant additive functor F.

(3) Let $R = \mathbb{Z}/p^n\mathbb{Z}$.
   (a) Show that R is an injective R-module.
   (b) Find a free resolution and an injective resolution for the cyclic module R/pR.

(4) (a) Let R be a left Noetherian ring. Show that every finitely generated R-module admits a free resolution in which every module is finitely generated.
   (b) Let D be a PID. Show that every finitely generated R-module admits a free resolution $F_\bullet$ with $F_i = 0$ for $i > 1$.

[1]Hint: You may want to consider left/right/short exact sequences involving C_i, $Z_i(C_\bullet)$, $B_i(C_\bullet)$, and $H_i(C_\bullet)$.