ASSIGNMENT #4

(1) Find two different composition series for the ring $R = \mathbb{Z}[x]/(6, x^2)$ as a module over itself.

(2) The purpose of this problem is to show that the ring

$$R = \begin{bmatrix} \mathbb{Z} & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix} = \left\{ \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \;\middle|\; a \in \mathbb{Z},\ b, c \in \mathbb{Q} \right\}$$

is left Noetherian but not right Noetherian.

(a) Let I be a left ideal. Show that if there is some $\begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \in I$ with $a \neq 0$, then for some ideal $J \subseteq \mathbb{Z}$,

$$I = \begin{bmatrix} J & 0 \\ \mathbb{Q} & 0 \end{bmatrix} \quad \text{or} \quad I = \begin{bmatrix} J & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}.$$

(b) Show that the left ideals contained in $\begin{bmatrix} 0 & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}$ are exactly the $\mathbb{Q}$-vector subspaces of $\begin{bmatrix} 0 & 0 \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}$. Conclude that R is left Noetherian.

(c) Show that for every $n \in \mathbb{N}$, the subset of R given by $\begin{bmatrix} 0 & 0 \\ \frac{1}{n}\mathbb{Z} & 0 \end{bmatrix}$ is a right ideal. Conclude that R is not right Noetherian.

(3) Computations with Schur's Lemma.

(a) Let D be a division ring and $n \geqslant 1$ be an integer. Show[1] that $\operatorname{End}_{\operatorname{Mat}_n(D)}(D^n) \cong D^{\mathrm{op}}$.

(b) Let $n \geqslant 3$, and let D_{2n} be the dihedral group of order $2n$. Show that the standard representation of D_{2n} on $V = \mathbb{R}^2$ acting by rotations and reflections is simple.

(c) With V as in the previous part, show that $\operatorname{End}_{\mathbb{R}[D_{2n}]}(V) \cong \mathbb{R}$ as rings.

(4) The goal of this problem is to find every irreducible $\mathbb{R}$-linear representation (up to isomorphism) of the dihedral group D_8 of order 8.

(a) Prove[2] there are 4 irreducible representations M_1, M_2, M_3, M_4 whose underlying $\mathbb{R}$-vector space is just $\mathbb{R}$ and that are pairwise nonisomorphic.

(b) Let M_5 be $\mathbb{R}^2$ with the standard action of D_8 (as in problem 3). Prove that $M_1, \ldots, M_5$ are the only irreducible $\mathbb{R}$-linear representations of D_8 up to isomorphism.

(5) Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ denote the group of quaternions.

(a) Find the Artin-Wedderburn decomposition of each of $\mathbb{C}[Q]$ and $\mathbb{C}[D_8]$ and use these to prove there is a ring isomorphism $\mathbb{C}[Q] \cong \mathbb{C}[D_8]$.

(b) Find[3] the Artin-Wedderburn decomposition of each of $\mathbb{R}[Q]$ and $\mathbb{R}[D_8]$ and use these to prove these rings are not isomorphic.

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[1] Hint: let $x \in D^{\mathrm{op}}$ correspond to the map given by multiplication on the right by x.

[2] Recall that a 1-dimensional representation of G is a homomorphism from G to $\mathbb{R}^\times$, and that any such map factors through the abelianization of G. It might be helpful to note that the derived subgroup of D_8 is $\langle r^2 \rangle$ where r is the "rotation" and the abelianization of D_8 is isomorphic to $C_2 \times C_2$.

[3] Hint: Use the previous problem to find the Artin-Wedderburn decomposition of $\mathbb{R}[D_8]$. To find the decomposition of $\mathbb{R}[Q]$, note that the group Q acts $\mathbb{R}$-linearly on the left on the division ring of real quaternions $\mathbb{H}$, via the identification of Q as the subgroup $\{\pm 1, \pm i, \pm j, \pm k\}$ of $\mathbb{H}^\times$, and thus $\mathbb{H}$ is a left $\mathbb{R}[Q]$-module. Thus, there is a surjective ring map $\mathbb{R}[Q] \to \mathbb{H}$.