ASSIGNMENT #1

(1) (a) Show that if R is a ring and $\alpha : M \to N$ is a morphism in $R-\mathbf{Mod}$, then α is monic if and only if it is injective, and α is epic if and only if it is surjective.

(b) Show that the map $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ in $\mathbb{Z}-\mathbf{Mod}$ has no left inverse (even though it is injective) and that the quotient map $\mathbb{Z} \twoheadrightarrow \mathbb{Z}/2\mathbb{Z}$ in $\mathbb{Z}-\mathbf{Mod}$ has no right inverse (even though it is surjective).

(2) (a) An abelian group M is *divisible* if for every $m \in M$ and nonzero $n \in \mathbb{Z}$, there is some $m' \in M$ such that $m = nm'$. Let $\mathbf{DAb}$ be the full subcategory of $\mathbf{Ab}$ consisting of all divisible abelian groups. Show that the quotient map $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is monic in $\mathbf{DAb}$ (even though it isn't injective).

(b) Show that the inclusion map $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic in $\mathbf{Ring}$ (even though it isn't surjective).

(3) (a) Find a pair of objects in $\mathbf{Fld}$ with no product.

(b) Find a pair of objects in $\mathbf{Fld}$ with no coproduct.

(c*) If K and L are fields of characteristic zero, do K and L admit a product/coproduct in $\mathbf{Fld}$?

(4) Let N be a left R-module, and $\{M_\lambda\}_{\lambda\in\Lambda}$ be a family of submodules of N. We say that N is the *internal direct sum* of $\{M_\lambda\}$ if the canonical map $\bigoplus_{\lambda\in\Lambda} M_\lambda \to N$ is an isomorphism. Show that N is the internal direct sum of $\{M_\lambda\}$ if and only if

- N is generated by $\bigcup_{\lambda\in\Lambda} M_\lambda$, and
- for every finite subset $\lambda_0, \lambda_1, \dots, \lambda_t$ of (at least two distinct) elements of Λ,

$$M_{\lambda_0} \cap (M_{\lambda_1} + \cdots + M_{\lambda_t}) = 0.$$

A *covariant functor* F between two categories $\mathscr{C}$ and $\mathscr{D}$ is a rule that assigns to each object A of $\mathscr{C}$ an object $F(A)$ of $\mathscr{D}$, and to each morphism $A \xrightarrow{\alpha} B$ of $\mathscr{C}$ a morphism $F(A) \xrightarrow{F(\alpha)} F(B)$ of $\mathscr{D}$ such that for every object A of $\mathscr{C}$, $F(1_A) = 1_{F(A)}$, and $F(\alpha \circ \beta) = F(\alpha) \circ F(\beta)$.

(5) Suppose that $\mathscr{C}$ and $\mathscr{D}$ are subcategories of $\mathbf{Set}$ (e.g., $\mathbf{Set}, \mathbf{Sgrp}, \mathbf{Grp}, \mathbf{Ab}, \mathbf{Ring}, R-\mathbf{Mod}, \mathbf{Top}$) and $F : \mathscr{C} \to \mathscr{D}$ is a covariant functor.

(a) Show that if α has a left inverse, then $F(\alpha)$ is injective (as a function).

(b) Show that if α has a right inverse, then $F(\alpha)$ is surjective (as a function).

(c) Use part (a) to show[1] that there is no covariant functor $F : \mathbf{Grp} \to \mathbf{Grp}$ that, on objects, maps a group to its center.

(d*) Show[1] that if we only assume that α is monic, then $F(\alpha)$ may not be injective.

(6) (a) Show[2] that in $\mathbb{Z}-\mathbf{Mod}$, the objects $\coprod_{n\in\mathbb{N}} \mathbb{Z}$ and $\prod_{n\in\mathbb{N}} \mathbb{Z}$ are not isomorphic.

(b*) Show[3] that $\prod_{n\in\mathbb{N}} \mathbb{Z}$ is not a free module.

[1] Hint: You might consider symmetric groups $\mathbb{S}_m \subseteq \mathbb{S}_n$.

[2] Hint: You may want to use the fact that the collection of finite sequences with values in a countable set is a countable set.

[3] Hint: Suppose so. Show that there is a countable free submodule T such that $\coprod_{n\in\mathbb{N}} \mathbb{Z} \subseteq T \subseteq \prod_{n\in\mathbb{N}} \mathbb{Z}$ and that the quotient $(\prod_{n\in\mathbb{N}} \mathbb{Z})/T$ is free. Then find a nonzero element in $(\prod_{n\in\mathbb{N}} \mathbb{Z})/T$ that is divisible by infinitely many integers.