Math 325. Exam #2

(1) Definitions/Theorem statements

(a) State the definition of *the limit of $f(x)$ as x approaches a*.

The limit of $f(x)$ as x approaches a is L provided:
for any $\varepsilon > 0$, there is some $\delta > 0$ such that
if $0 < |x-a| < \delta$, then $f(x)$ is defined and
$|f(x) - L| < \varepsilon$.

(b) State the *Extreme Value Theorem*.

If f is continuous on the closed interval $[a,b]$,
then there exist $r, s \in [a,b]$ such that
$f(r) \leq f(x) \leq f(s)$ for all $x \in [a,b]$.

(c) State the *Bolzano-Weierstrass Theorem*.

Every sequence has a monotone subsequence.

(2) Determine if each of the following statements is TRUE or FALSE, and justify your choice with a short argument or a counterexample.

(a) There is some $t \in [-1, 1]$ such that $t^4 + 5t = -1$.

True

Since $f(x) = x^4 + 5x$ is a polynomial, and $f(-1) = -4 \leq -1 \leq 6 = f(1)$, the Intermediate value Theorem ensures such a value of t.

(b) Every sequence has a convergent subsequence.

FALSE

Every subsequence of $\{n\}_{n=1}^{\infty}$ diverges, since any subsequence is not bounded above.

(c) If f and g are functions such that $\lim_{x\to 5} f(x)$ and $\lim_{x\to 5} g(x)$ both exist, then $\lim_{x\to 5} \frac{f(x)}{g(x)}$ exists.

FALSE

Let $f(x)=1$ and $g(x)=x-5$.

then $\lim_{x\to 5} f(x)=1$, $\lim_{x\to 5} g(x)=0$,

~~but~~ $\lim_{x\to 5} \frac{f(x)}{g(x)}$ does not exist.

(d) If $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, and $\lim_{k\to\infty} a_{2k} = 9$, then $\lim_{k\to\infty} a_{2k-1} = 9$.

TRUE

Since $\{a_n\}_{n=1}^{\infty}$ is Cauchy, it is convergent, ~~so~~ every subsequence must converge to the same value.

(e) If $\lim_{x\to 0} f(x) = 3$, then the sequence $\{a_n\}_{n=1}^{\infty}$ converges to 0, where $a_n = \dfrac{f(1/n)}{n}$.

TRUE

Since $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0 (and no value is 0), it follows that $\{f(1/n)\}_{n=1}^{\infty}$ converges to 3.

Then we can consider $\dfrac{f(1/n)}{n}$ as $f(1/n) \cdot \frac{1}{n}$, and since $\{f(1/n)\}_{n=1}^{\infty}$ converges to 3 & $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges to 0, the sequence converges to the product, 0.

(3) Proofs.

(a) Prove that the function

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \geq 1 \\ x & \text{if } x < 1 \end{cases}$$

is continuous at the point $x = 1$.

Let $\varepsilon > 0$. Take $\delta = \varepsilon/2$.

Let x be a real number such that $|x - 1| < \delta$.

If $x \geq 1$, then

$|f(x) - f(1)| = |(2x-1) - 1| = |2x - 2|$

$= 2|x - 1| < 2\delta = \varepsilon$.

If $x < 1$, then

$|f(x) - f(1)| = |x - 1| < \delta = \varepsilon/2 < \varepsilon$.

Thus, for all such x,

$|f(x) - f(1)| < \varepsilon$.

This shows that f is continuous at $x = 1$. $\boxtimes$

(b) Assume f is a function whose domain is all of $\mathbb{R}$, let a be any real number, and assume that $\lim_{x\to 2} f(x) = L$ for some real number L. Prove that[1] if $f(x) \le a$ for all x, then $L \le a$.

By way of contradiction, suppose that $L > a$. Taking $\varepsilon = L - a$, which is positive by assumption, there is some $\delta > 0$ such that

$$|f(x) - L| < L - a \quad \text{for all}$$

x such that $0 < |x - a| < \delta$.

Thus, for such x,

$$f(x) - L > -(L-a) = a - L,$$

so $f(x) > a$, which contradicts our hypothesis. We conclude that $L \le a$.

$\boxtimes$

[1] Hint: I recommend a proof by contradiction.

Bonus: TRUE or FALSE: There is a sequence $\{a_n\}_{n=1}^{\infty}$ such that

$$\{x \in \mathbb{R} \mid \text{there is a subsequence of } \{a_n\}_{n=1}^{\infty} \text{ that converges to } x\} = [0, 7].$$

TRUE

Recall that there is a sequence of rational numbers $\{q_n\}_{n=1}^{\infty}$ in which every rational number occurs infinitely many times. Let $\{q_{n_k}\}_{k=1}^{\infty}$ be the subsequence of $\{q_n\}_{n=1}^{\infty}$ obtained by skipping all terms that are not in the interval $[0,7]$. Call this sequence $\{r_n\}_{n=1}^{\infty}$.

Let $\{r_{n_k}\}_{k=1}^{\infty}$ be a convergent subsequence of $\{r_n\}_{n=1}^{\infty}$. Since $0 \le r_{n_k} \le 7$ for all k, we must have $0 \le \lim_{k\to\infty} r_{n_k} \le 7$.

On the other hand, let $a \in [0,7]$.

Since 0 occurs in $\{r_n\}_{n=1}^{\infty}$ infinitely many times, there is a constant subsequence $\{0\}$ of $\{r_n\}_{n=1}^{\infty}$, which converges to 0.

If $a \in (0,7]$, note that there is a sequence of rational numbers $\{v_n\}_{n=1}^{\infty}$ that converges to a.

Passing to a subsequence, we can assume it consists of positive numbers (i.e., let $\varepsilon = a$, and take the subsequence $\{v_{N+n}\}_{n=1}^{\infty}$, with N as in definition of converges).

This sequence is now a subsequence of $\{r_n\}_{n=1}^{\infty}$, so a is a limit of a subsequence of $\{r_n\}_{n=1}^{\infty}$. $\boxtimes$