Math 325. Exam #1

(1) Definitions/Theorems

(a) State the definition for a real number b to be a *lower bound* for a set S of real numbers.

For every $x \in S$, $b \leq x$.

(b) State the *Completeness Axiom* for $\mathbb{R}$.

Every nonempty bounded above subset of $\mathbb{R}$ has a supremem.

(c) State the definition for a sequence $\{a_n\}_{n=1}^{\infty}$ to *diverge to* $-\infty$.

For every $m \in \mathbb{R}$, there is some $N \in \mathbb{R}$ such that for all natural numbers $n > N$, $a_n < m$.

(2) Short answer.

(a) Write the negation of the following statement in its simplest form:

For every $x \in S$, there exists a natural number integer j such that $x < j + 3$.

There exists $x \in S$ such that for every integer j, $x \geq j + 3$.

(b) Write the contrapositive of the following statement in its simplest form:

If $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both diverge, then $\{a_n+b_n\}_{n=1}^{\infty}$ diverges.

If $\{a_n+b_n\}_{n=1}^{\infty}$ converges, then $\{a_n\}_{n=1}^{\infty}$ converges or $\{b_n\}_{n=1}^{\infty}$ converges.

(3) Determine if each of the following statements is TRUE or FALSE, and justify your choice with a short argument or a counterexample.

(a) Every convergent sequence is a monotone sequence.

False. For example, take $\{\frac{(-1)^n}{n}\}_{n=1}^{\infty}$.

(b) If $a_n^2 < 4$ and $a_{n+1} < a_n$ for all $n \in \mathbb{N}$, then $\{a_n\}_{n=1}^{\infty}$ converges.

True. We have $-2 < a_n < 2$ for all n, so the sequence is bounded. From the hypotheses, it is decreasing, so it converges.

(c) For any open interval $S = (a, b)$ with $a < b$, there is no smallest irrational number in S.

True. Suppose there was an open interval (a, b) with a smallest irrational number z. By Density of Irrationals, there is some irrational number y such that $a < y < z$. Then y is a smaller irrational number in (a, b), contradicting the existence of z.

(d) There are sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ such that

- $\{a_n\}_{n=1}^{\infty}$ converges,
- $\{b_n\}_{n=1}^{\infty}$ diverges, and
- $\{a_n + b_n\}_{n=1}^{\infty}$ converges.

False. If $\{a_n\}_{n=1}^{\infty}$ converges to L and $\{a_n+b_n\}_{n=1}^{\infty}$ converges to M, then by our Theorem on limits and algebra, $\{b_n\}_{n=1}^{\infty}$ converges to $M - L$.

(4) Proofs.

(a) Use the formal definition (and not any theorems) of a sequence to converge to prove that $\left\{2 + \frac{(-1)^n}{\sqrt{n}}\right\}_{n=1}^{\infty}$ converges to 2.

Let $\varepsilon > 0$ be arbitrary. Take $N = \frac{1}{\varepsilon^2}$. Observe that $\varepsilon^2 = \frac{1}{N}$, so $\varepsilon = \frac{1}{\sqrt{N}}$. Let n be an arbitrary natural number larger than N. Then

$$\left|2 + \frac{(-1)^n}{\sqrt{n}} - 2\right| = \left|\frac{(-1)^n}{\sqrt{n}}\right| = \frac{1}{\sqrt{n}} < \frac{1}{\sqrt{N}} = \varepsilon.$$

This shows that $\left\{2 + \frac{(-1)^n}{\sqrt{n}}\right\}_{n=1}^{\infty}$ converges to 2.

(b) Let S be a nonempty bounded above subset of $\mathbb{R}$, and let $\ell = \sup(S)$. Define $T = \{3s \mid s \in S\}$. Prove that $\sup(T) = 3\ell$.

First, we show that 3ℓ is an upper bound for T. Let $t \in T$. We can write $t = 3s$ for some $s \in S$. Since $s \leq \ell$, we have $t = 3s \leq 3\ell$, so 3ℓ is indeed an upper bound.

Next, we show that if b is any upper bound for T, then $b \geq \ell$. Let b be an upper bound for T. This means that $b \geq t$ for any $t \in T$. Note that $b/3$ is an upper bound for S: if $s \in S$, then $t = 3s \in T$, so $3s = t \leq b$, so $s \leq b/3$. By definition of supremum, $b/3 \geq \ell$, but then $b \geq 3\ell$, as required.

Bonus: Prove or disprove: Every convergent sequence of integers converges to an integer.

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of integers, and suppose that $\{a_n\}_{n=1}^{\infty}$ converges to L. There is a unique integer m such that $m \leq L < m+1$ (by a Theorem in class, or by common arithmetic knowledge). Suppose that L is not an integer. Then $m < L < m+1$. Note that there is no integer t such that $m < t < m+1$. Take $\varepsilon = \min\{L-m, m+1-L\}$, which is strictly positive. There exists N such that for all natural numbers $n > N$ we have $|a_n - L| < \varepsilon$. For any such n, we have

$$a_n - L < \varepsilon \leq m + 1 - L, \quad \text{so } a_n < m + 1$$

and

$$a_n - L > -\varepsilon \geq -(L - m), \quad \text{so } a_n > m.$$

But this contradicts that there is no integer t such that $m < t < m+1$! This contradiction shows that L must be an integer.