Math 325-002 — Problem Set #5
Due: Wednesday, September 29 by 5pm

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

If you do work with others, I ask that you write something along the top like "I collaborated with Steven Smale on problems 1 and 3". If you use a reference, indicate so clearly in your solutions. In short, be intellectually honest at all times.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Prove that the sequence
$$\left\{\frac{8n^2-5n+3}{4n^2+1}\right\}_{n=1}^{\infty}$$
converges. (This includes finding to what it converges.) You should use Theorem 12.2, but carefully justify every step using the Theorem.

(2) Prove that the sequence $\{\sqrt{n}\}_{n=1}^{\infty}$ diverges.

(3) Assume that $\{a_n\}_{n=1}^{\infty}$ converges to zero, and that $a_n \geq 0$ for all natural numbers n. Prove that $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to zero also.

(4) Assume that $\{a_n\}_{n=1}^{\infty}$ converges to L, and that $a_n \geq 0$ for all natural numbers n. Prove[1] that $\{\sqrt{a_n}\}_{n=1}^{\infty}$ converges to $\sqrt{L}$.

(5) Let $\{a_n\}_{n=1}^{\infty}$ converge to zero, and let $\{b_n\}_{n=1}^{\infty}$ be any bounded sequence (that may or may not be convergent). Prove that $\{a_nb_n\}_{n=1}^{\infty}$ converges to zero.

(6) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence such that $a_n > 0$ for all n. Show that $\{a_n\}_{n=1}^{\infty}$ diverges to ∞ if and only if $\{\frac{1}{a_n}\}_{n=1}^{\infty}$ converges to 0.

[1]Use an old HW problem to explain why $L \geq 0$. Use the previous problem to deal with the case $L = 0$. For $L > 0$, use $(\sqrt{a_n}-\sqrt{L})(\sqrt{a_n}+\sqrt{L}) = a_n - L$ to deduce that
$$\sqrt{a_n}-\sqrt{L} \leq \frac{|a_n-L|}{\sqrt{L}}.$$