Math 325-002 — Problem Set #8
Due: Wednesday, November 3 by 5 pm

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Using just the $\epsilon - \delta$ definition of limit, show that

$$\lim_{x \to -1} \frac{x^2 - 6x - 7}{x + 1} = -8.$$

(2) Using just the $\epsilon - \delta$ definition of limit, show that[1] for any $a > 0$, $\lim_{x \to a} \sqrt{x} = \sqrt{a}$.

(3) Let $f(x)$ be the function with domain $\mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Z}, \text{ and} \\ -1 & \text{if } x \notin \mathbb{Z}. \end{cases}$$

Prove that for any $a \in \mathbb{R}$, we have $\lim_{x \to a} f(x) = -1$.

(4) Let $f(x)$ be the function with domain $\mathbb{R}$ given by the rule

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \text{ and} \\ -1 & \text{otherwise.} \end{cases}$$

Prove that $\lim_{x \to 0} f(x)$ does not exist.

(5) Let $f(x)$ be the function with domain $\mathbb{R}$ given by the rule

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \text{ and} \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

(a) Prove that $\lim_{x \to 0} f(x) = 0$.
(b) Prove that if $a \neq 0$, then $\lim_{x \to a} f(x)$ does not exist.

Definition: Let f be a function and $a \in \mathbb{R}$. We say that *the limit of $f(x)$ as x approaches a from the right is L* provided:

> For every $\epsilon > 0$, there is some $\delta > 0$ such that for all x satisfying $a < x < a + \delta$, we have that f is defined at x and also that $|f(x) - L| < \epsilon$.

In this case, we write $\displaystyle\lim_{x \to a^+} f(x) = L$.

(6) Use the definition to prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

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[1] Hint: You may want to use that $|\sqrt{x} - \sqrt{a}| = \dfrac{|x - a|}{|\sqrt{x} + \sqrt{a}|} \leq \dfrac{|x - a|}{|\sqrt{a}|}$.