Math 325-002 — Problem Set #7
Due: Friday, October 22 by 5 pm

Instructions: You are encouraged to work together on these problems, but each student should hand in their own final draft, written in a way that indicates their individual understanding of the solutions. Never submit something for grading that you do not completely understand.

Please write neatly, using complete sentences and correct punctuation. Label the problems clearly.

(1) Define a sequence $\{a_n\}_{n=1}^{\infty}$ recursively by $a_1 = 2$ and $a_n = \frac{a_{n-1}}{2} + \frac{1}{a_{n-1}}$ for all $n \geq 2$.
   (a) Prove $a_n > 0$ for all $n \in \mathbb{N}$ by induction on n.
   (b) Prove[1] $a_n^2 \geq 2$ for all $n \in \mathbb{N}$.
   (c) Prove[2] the sequence is decreasing.
   (d) Since the sequence is decreasing and bounded below, it necessarily converges. Determine what the sequence converges to.[3]

(2) For each of the following, give an explicit example as indicated. No proofs are necessary.
   (a) A sequence that has a subsequence that converge to 1, another subsequence that converges to 2, and a third subsequence that converges to 3.
   (b) A sequence that has one subsequence that is monotone and converges to 0 and another subsequence that is monotone and diverges to $+\infty$.
   (c) A sequence of natural numbers such that for each $j \in \mathbb{N}$, it has a subsequence that converges to j. (Feel free to just describe the pattern – no formulas needed. As a hint, recall that the constant sequence j converges to j.)

(3) Prove that if $\{a_n\}_{n=1}^{\infty}$ is a sequence that diverges to ∞, then every subsequence of $\{a_n\}_{n=1}^{\infty}$ diverges to ∞.

(4) Prove that for every real number x, there is a sequence of irrational numbers that converges to x.

(5) Let $\{a_n\}_{n=1}^{\infty}$ be any sequence and L any real number. Prove that if $\{a_n\}_{n=1}^{\infty}$ does not converge to L, then there exists an $\epsilon > 0$ and a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ such that $|a_{n_k} - L| \geq \epsilon$ for all k.

(6) Determine whether each of the following sequences is Cauchy, and prove your answer just using the definition of Cauchy (not any theorems).
   (a) $\{\sqrt{n}\}_{n=1}^{\infty}$
   (b) $\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty}$

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[1]Hint: Write $a_n - 2$ in terms of a_{n-1}, and factor the expression.

[2]Hint: Consider $a_n - a_{n-1}$ and use (b).

[3]Hint: Use that

$$\lim_{n\to\infty} a_n = \frac{\lim_{n\to\infty} a_{n-1}}{2} + \frac{1}{\lim_{n\to\infty} a_{n-1}}$$

so that if we set $L = \lim_{n\to\infty} a_n$ then we have $L = \frac{L}{2} + \frac{1}{L}$.