Math 314. Week 9 worksheet (§4.7, §5.1, §5.2).

An **eigenvector** of an $n \times n$ matrix A is a *nonzero* vector $\mathbf{v}$ such that $\boxed{A\mathbf{v} = \lambda\mathbf{v}}$ for some scalar λ; the number λ is the **eigenvalue** associated to $\mathbf{v}$. We can think of this as saying that $A\mathbf{v}$ goes in the same (or backwards) direction as $\mathbf{v}$, with different magnitude, and λ is the factor by which the magnitude changes.

We say that λ is an **eigenvalue** of A if it is the eigenvalue associated to some eigenvector of A: that is, λ is an eigenvalue of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$.

A. Geometric examples of eigenvectors. For each of the following 2×2 matrices, use the description of its associated linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ to find eigenvectors and eigenvalues for the matrix.

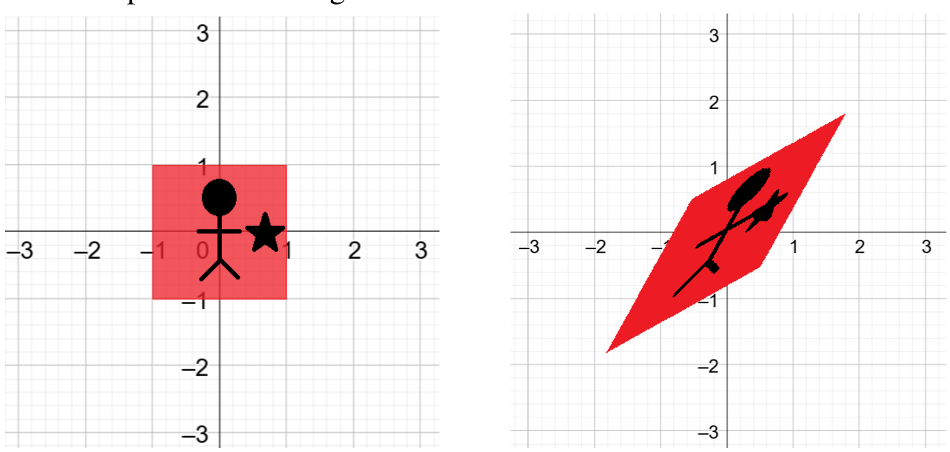
(1) $A = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$: shrink by $1/2$ in the horizontal direction.

(2) $B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$: reflect over the line $y = -x$.

(3) $C = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$: rotate by $45°$ counterclockwise.

(4) $D = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$: horizontal shear.

B. Eigenvectors from pictures. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that transforms the picture on the left to the picture on the right:



Without computing the standard matrix of T, use the picture to find eigenvectors and eigenvalues for it.

The λ **eigenspace** of an $n \times n$ matrix A is $\mathrm{Null}(A - \lambda I)$, where I is the $n \times n$ identity matrix. This consists of the zero vector $\mathbf{0}$ and all of the eigenvectors for A with eigenvalue λ. If λ is not an eigenvalue of A, then this is just $\{\mathbf{0}\}$.

The **characteristic polynomial** of A is $\det(A - \lambda I)$, considered as a polynomial in the variable λ. Its roots are the eigenvalues of A. The **multiplicity**[1] of an eigenvalue λ of A is its multiplicity as a root of the characterstic polynomial.

[1] This is also called algebraic multiplicity.

C. COMPUTING EIGENVALUES AND EIGENVECTORS. Let $A = \begin{bmatrix} 3 & -4 \\ -5 & 2 \end{bmatrix}$.

(1) Compute the characteristic polynomial of A.
(2) Find the roots of the characteristic polynomial of A. *These are the eigenvalues of* A.
(3) Pick one of your eigenvalues, maybe call it λ_1, and compute the λ_1 eigenspace of A.
(4) Take the other eigenvalue, maybe call it λ_2, and compute the λ_2 eigenspace of A.

D. EIGENVECTORS OF DIAGONAL MATRICES AND TRIANGULAR MATRICES.

(1) If a, b, c are three different numbers, find the eigenvalues and eigenvectors of $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$. What are the multiplicities of the eigenvalues? What are the dimensions of the eigenspaces?
(2) If a, b are two different numbers, find the eigenvalues and eigenvectors of $B = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$. What are the multiplicities of the eigenvalues? What are the dimensions of the eigenspaces?
(3) If a, b are two different numbers, find the eigenvalues and eigenvectors of $C = \begin{bmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}$. What are the multiplicities of the eigenvalues? What are the dimensions of the eigenspaces?

If V and W are vector spaces, a function $T : V \to W$ is a **linear transformation** if

- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$, and
- $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $c \in \mathbb{R}, \mathbf{v} \in V$.

The **kernel** of a linear transformation $T : V \to W$ is the set of $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{0}$ (the zero vector in W).

The **range** of a linear transformation $T : V \to W$ is the set of all outputs $\mathbf{w} = T(\mathbf{v})$ in W, for all possible inputs $\mathbf{v} \in V$.

E. A LINEAR TRANSFORMATION ON POLYNOMIALS. Let P be the vector space of all polynomials (of any degree). Consider the function $D : P \to P$ given by $D(f(t)) = \frac{df}{dt}$.

(1) Explain why D is a linear transformation.
(2) What is the kernel of D?
(3) What is the range of D?
(4) Is the function $S : P \to P$ given by $S(f(t)) = f(t)^2$ a linear transformation?

F. A LINEAR TRANSFORMATION ON MATRICES. Consider the vector space $M_{2\times 2}$ of 2×2 matrices. Let $A = \begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$.

(1) Show that the function $F : M_{2\times 2} \to M_{2\times 2}$ given by $F(X) = AX$ is a linear transformation.
(2*) What is the kernel of F? Can you find a basis for it?
(3*) What is the range of F? Can you find a basis for it?

If V is a vector space, and $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ and $\mathcal{C} = \{\mathbf{c_1}, \ldots, \mathbf{c_n}\}$ are two bases for V, then the $\mathcal{B}$-coordinates and the $\mathcal{C}$-coordinates of any vector are related by the formula

$$[\mathbf{v}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}} \cdot [\mathbf{v}]_{\mathcal{B}}, \qquad \text{where } P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} [\mathbf{b_1}]_{\mathcal{C}} & \cdots & [\mathbf{b_n}]_{\mathcal{C}} \end{bmatrix}.$$

G. DIFFERENT COORDINATES FOR P_3. Consider the two bases $\mathcal{B} = \{t^3, t^2, t, 1\}$ and $\mathcal{C} = \{t^3 - 1, t^2 - 1, t - 1, 1\}$ for P_3, the vector space of polynomials of degree at most 3. Find the matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$, and use this to compute $[3t^3 - 4t^2 + t - 5]_{\mathcal{C}}$.

H. COMPARE WITH SECTION 4.4.

(1) Convince yourself that the "usual coordinates" on $\mathbb{R}^n$ are the same as $\mathcal{E}$-coordinates, where $\mathcal{E} = \{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$.
(2) We have seen the matrix $P_{\mathcal{E}\leftarrow\mathcal{B}}$ with a different name in §4.4. What was it?
(3) If $\mathcal{B}$ and $\mathcal{C}$ are two different bases for a vector space V, how are $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P_{\mathcal{B}\leftarrow\mathcal{C}}$ related?
(4) If $\mathcal{B}$ and $\mathcal{C}$ are two different bases for $\mathbb{R}^n$, then how are the matrices $P_{\mathcal{C}\leftarrow\mathcal{B}}$, $P_{\mathcal{C}}$, and $P_{\mathcal{B}}$ related?

I*. SOMETHING ABOUT RANK. Let A be an $m \times n$ matrix, and B be an $n \times k$ matrix.

(1) Explain why $\mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$.
(2) Explain why $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.
(3) Explain why $\mathrm{Null}(B) \subseteq \mathrm{Null}(A)$.
(4) Explain why $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$.
(5) Is $\mathrm{rank}(AB) = \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$?

J*. SOMETHING ELSE ABOUT BASES.

(1) A set of vectors S in a vector space V is a *maximal linearly independent set* if S is linearly independent, and $S \cup \{\mathbf{v}\}$ is linearly dependent for all $\mathbf{v} \notin V$. Explain why a maximal linearly independent set in V is a basis for V.
(2) A set of vectors S in a vector space V is a *minimal spanning set* if S spans V, and $S \setminus \{\mathbf{s}\}$ does not span V for all $\mathbf{s} \in S$. Explain why a minimal spanning set in V is a basis for V.