4.1) $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 3 \end{bmatrix}$

2) $7t^3 - t^2 + \pi$

3) If $p(t) = at^3 + bt^2 + ct + d \in P_3$,

then $p(t) = a(t^3-1) + b(t^2-1) + c(t-1) + (a+b+c+d)\cdot 1$

~~so~~ it spans.

If $a(t^3-1) + b(t^2-1) + c(t-1) + d = 0$ (zero polynomial),

then $a = b = c = d = 0$, so it's LI.

4) $7(t^3-1) - (t^2-1) + \pi$

$= 7t^3 - t^2 + (\pi - 6)$

5) $\begin{bmatrix} 2 \\ -1 \\ 0 \\ 4 \end{bmatrix}$

B. 1) $\begin{bmatrix} 3 & 7 & -5 \end{bmatrix}$

2) It is a null space.

3) $\left\{ \begin{bmatrix} -7/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5/3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

4) $\begin{bmatrix} -2/3 \\ 1 \\ 1 \end{bmatrix}$

different answers for different bases in (3)

5) $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$

6) No, it's ~~not~~ in H

C. 1) no; not a matrix

2) $\begin{bmatrix} 7/2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$

3) no; ~~not~~ a set of vectors

4) yes

5) $A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$

6) If $[v]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, $\underline{v} = c_1\underline{b_1} + c_2\underline{b_2} + c_3\underline{b_3}$,

so $A[v]_B = c_1\underline{b_1} + c_2\underline{b_2} + c_3\underline{b_3} = \underline{v}$.

7) $\begin{bmatrix} 6 \\ -2 \\ -3 \end{bmatrix}$

8) $\begin{bmatrix} 2 \\ -9 \\ 9 \end{bmatrix}$.

D. omitted; see (C) above.

E. dim Col $A = 3$; dim Null $A = 2$.

F. 1) 4   5) linearly dependent
2) 2   6) no
3) no   7) 4
4) 3   8) <4

G. Given a finite set of polynomials, they all have degree $\leq N$ for some N. Then the span is contained in $\mathbb{P}_N$, so t^{N+1} is not in the span. Thus, they can't span $\mathbb{P}$.

$\{1, t, t^2, t^3, t^4, \dots\}$ is a basis.

H. 1) Because the $\underline{v}$'s span.

2) More columns than rows $\Rightarrow$ homog equation has a solution.

3) $\sum_{j=1}^{n} b_j \underline{w}_j = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} b_j \underline{v}_i$

$$= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} b_j \right) \underline{v}_i = \underline{0}.$$

4) Yup.

5) 1) If $\beta, \mathcal{C}$ are bases, β is LI & $\mathcal{C}$ spans so $\#\beta \leq \#\mathcal{C}$. Likewise, β spans & $\mathcal{C}$ LI so $\#\beta \geq \#\mathcal{C}$. Must be equal.

2) A basis is LI, so clear.

3) A basis spans, so clear.

6) A basis for H is a LI subset of V, and a basis for V spans V, so it follows.

I. 1) $Col(AB) = \{ AB\underline{x} \mid \underline{x} \in \mathbb{R}^k \}$

$Col(A) = \{ A\underline{y} \mid \underline{y} \in \mathbb{R}^m \}$.

$AB\underline{x} = A(B\underline{x}), \quad B\underline{x} \in \mathbb{R}^m$.

2) Follows from (1) and dim of a subspace $\leq$ dim of a space.

3) Should say $Null(B) \leq Null(AB)$:

If $\underline{x} \in Null\, B$, $B\underline{x} = 0$, so $AB\underline{x} = \cancel{A0} = 0$

4) $\dim Null(B) \leq \dim Null(AB)$

$\Rightarrow k - \dim Null(B) \geq k - \dim Null(AB)$

$\Rightarrow rank(B) \geq rank(AB)$.

5) Not always; e.g. $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \ldots$

J. 1) If S LI, $S \cup \{\underline{v}\}$ LD, then $\sum a_i s_i + b\underline{v} = 0$ some $\underline{s_i} \in S$, $\{a_i\} \cup \{b\}$ not all zero. If $b = 0$, then S is LD, so $b \neq 0$. Thus, we can solve for $\underline{v}$ as a lin. combo. of $\{s_i\}$, so $\underline{v} \in \operatorname{span} S$.

2) If S spans V, and S is LD, we can write $\sum a_i \underline{s_i} = 0$ some $\underline{s_i} \in S$, $\{a_i\}$ not all zero. Then, we can solve for some $\underline{s_i}$ as a lin. combo. of the others. Thus any linear combo of S can be written as a linear combo of $S \setminus \{\underline{s_i}\}$ by substituting, so $S \setminus \{\underline{s_i}\}$ spans V.