Math 314. Week 8 worksheet (§4.4, §4.5, §4.6).

A set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ in a vector space (or subspace) V is a **basis** for V if it spans V and it is linearly independent.
If $\mathcal{B} = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$ is a basis for V, then every element of V can be written as a linear combination of these vectors

$$\mathbf{v} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}$$

in exactly one way. We say that the stack of numbers

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the vector of $\mathcal{B}$-coordinates of $\mathbf{v}$.

A. COORDINATES FOR P_3

(1) The set $\mathcal{B} = \{t^3, t^2, t, 1\}$ is a basis for P_3, the vector space of polynomials of degree at most 3. Find the $\mathcal{B}$-coordinates of the polynomial $p(t) = 2^3 - t^2 + 3$.

(2) If $[q(t)]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -1 \\ 0 \\ \pi \end{bmatrix}$, what is $q(t)$?

(3) Show that the set of polynomials $\mathcal{C} = \{t^3 - 1, t^2 - 1, t - 1, 1\}$ is also a basis for P_3.

(4) Find $q(t)$ where $[q(t)]_{\mathcal{C}} = \begin{bmatrix} 7 \\ -1 \\ 0 \\ \pi \end{bmatrix}$.

(5) Find $[2^3 - t^2 + 3]_{\mathcal{C}}$.

B. COORDINATES FOR A SUBSPACE OF $\mathbb{R}^3$. Consider the plane H given by the equation $3x+7y-5z = 0$ in $\mathbb{R}^3$.

(1) H is the null space of a matrix—which matrix?
(2) Explain in five words or less why H is a subspace of $\mathbb{R}^3$.
(3) Find a basis $\mathcal{B}$ for H.
(4) Using the basis you found, determine the point on H with $\mathcal{B}$-coordinates $[1, 1]^T$.
(5) Using the basis you found, determine the $\mathcal{B}$-coordinates of $[-1, -1, -2]^T$.
(6) Using the basis you found, can you find the $\mathcal{B}$-coordinates of $[0, -1, -2]^T$?

C. DIFFERENT[1] COORDINATES ON $\mathbb{R}^3$. Let

$$\mathbf{b_1} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \ \mathbf{b_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{b_3} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \ A = \begin{bmatrix} \mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3} \end{bmatrix}, \text{ and } \mathcal{C} = \{\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3}\}.$$

(1) Is $\mathcal{C}$ invertible? If so, find the inverse of $\mathcal{C}$.
(2) Is A invertible? If so, find the inverse of A.
(3) Is A a basis for $\mathbb{R}^3$?

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[1] Trick questions ahead!

(4) Is $\mathcal{C}$ a basis for $\mathbb{R}^3$?
(5) Rewrite the expression $c_1\mathbf{b_1} + c_2\mathbf{b_2} + c_3\mathbf{b_3}$ as a product involving the matrix A and a vector.
(6) Explain why $A[\mathbf{v}]_{\mathcal{C}} = \mathbf{v}$ for all vectors $\mathbf{v} \in \mathbb{R}^3$.
(7) If $[\mathbf{v}]_{\mathcal{C}} = [3, -1, -1]^T$, then use the previous part to find $\mathbf{v}$.
(8) Use part (2) to find the $\mathcal{C}$-coordinates of the vector $[4, 0, 9]^T$.

D. CHANGE-OF-COORDINATE MATRIX. If $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is a basis for $\mathbb{R}^n$, then $P_{\mathcal{B}} = [\mathbf{b_1} \cdots \mathbf{b_n}]$ is called the $\mathcal{B}$ change-of-coordinates matrix. Explain why $P_{\mathcal{B}} \cdot [\mathbf{x}]_{\mathcal{B}} = \S$ and $P_{\mathcal{B}}^{-1} \cdot \mathbf{x} = [\mathbf{x}]_{\mathcal{B}}$ for all $\mathbf{x} \in \mathbb{R}^n$.

DEFINITION: The **dimension** of a vector space V is the number[2] of vectors in any basis for V.

FACTS ABOUT DIMENSION: For an n-dimensional vector space V,
(1) The number of vectors in any basis for V is exactly n.
(2) Any set of vectors that spans V has at least n elements.
(3) Any linearly independent set of vectors has at most n elements.
(4) If a set of n vectors is either linearly independent OR spans V, then it does both.

The dimension of the column space of a matrix is called its **rank**. This number is equal to the number of pivots. Furthermore,

$$\text{rank}(A) + \dim \text{Null}(A) = \#\text{ columns of } A.$$

E. DIMENSIONS OF COLUMN SPACES AND NULL SPACES. Find the dimension of $\text{Col}(A)$ and $\text{Null}(A)$, where $A = \begin{bmatrix} 1 & -7 & 8 & 1 & 5 \\ 0 & 1 & 3 & 0 & 8 \\ 0 & 0 & 0 & 7 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

F. USING THE RANK-NULLITY THEOREM. Let A be a 4×7 matrix.
(1) If the rank of A is 3, what is the dimension of $\text{Null}(A)$?
(2) If the dimension of $\text{Null}(A)$ is 5, what is the rank of A?
(3) Can $\dim \text{Null}(A) = 2$? Why or why not?
(4) Suppose that $A\mathbf{x} = \mathbf{b}$ always has a solution. What is $\dim \text{Null}(A)$?
(5) If the rank of A is three, then any set of four of the columns of A is [WHAT]?
(6) If the rank of A is three, then can a set of three of vectors span the null space of A?
(7) Suppose that the solution set to $A\mathbf{x} = [3, -1, 2, 6]^T$ is, in parametric vector form,

$$[1, 0, 1, 2, 8, 9, \pi]^T + r[e, \sqrt{2}, 0, 1, 2, 3, 4]^T + s[5, 5, 5, 0, 1, 9, 9]^T + t[8, 6, 7, 5, 3, 0, 9]^T, r, s, t \in \mathbb{R}.$$

What can you say about the rank of A?
(8) Suppose that the solution set to $A\mathbf{x} = [3, -1, 2, 6]^T$ is $\varnothing$. What can you say about the rank of A?

G. AN INFINITE DIMENSIONAL VECTOR SPACE. Explain why no finite set of polynomials spans the vector space P of polynomials (of any degree). Conclude that P is infinite-dimensional. Now find a basis for P.

[2]...*if* there is a finite set of vectors that is a basis for V. Otherwise, we say V is **infinite-dimensional**. Also, the dimension of the vector space $\{\mathbf{0}\}$ is 0.

H*. EVERYTHING ABOUT DIMENSION, ALMOST. Let V be a vector space. Suppose that the set $\{\mathbf{v_1}, \dots, \mathbf{v_m}\}$ spans V, and $\{\mathbf{w_1}, \dots, \mathbf{w_n}\}$ is another set of vectors in V.

(1) Explain why there are a bunch of numbers a_{ij}, where $i = 1, \dots, m$, $j = 1, \dots, n$, such that

$$\begin{aligned} \mathbf{w_1} &= a_{11}\mathbf{v_1} + \cdots + a_{m1}\mathbf{v_m} \\ \vdots \quad & \quad \vdots \qquad\qquad\quad \vdots \\ \mathbf{w_n} &= a_{1n}\mathbf{v_1} + \cdots + a_{mn}\mathbf{v_m} \end{aligned}$$

In short, $\mathbf{w_j} = \sum_{i=1}^{m} a_{ij}\mathbf{v_i}$ for every j.

(2) Put all these numbers a_{ij} into a matrix[3] $A = [a_{ij}]$. If $n > m$, explain why there is a nonzero vector $\mathbf{b} = [b_1, \dots, b_n]^T$ in the null space of A. In short, $\sum_{j=1}^{n} a_{ij}b_j = 0$ for every i.

(3) Continuing the story from the previous part, show that $\sum_{j=1}^{n} b_j\mathbf{w_j} = \mathbf{0}$.

(4) Conclude that S is *any* spanning set in V, and T is *any* linearly independent set on V, then S has at least as many elements as T.

(5) Use the last observation to justify facts (1)–(3) about dimension.

(6) Use part (4) to explain why $\dim(H) \leq \dim(V)$ for a subspace $H \subseteq V$.

I*. SOMETHING ABOUT RANK. Let A be an $m \times n$ matrix, and B be an $n \times k$ matrix.

(1) Explain why $\mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$.

(2) Explain why $\mathrm{rank}(AB) \leq \mathrm{rank}(A)$.

(3) Explain why $\mathrm{Null}(B) \subseteq \mathrm{Null}(A)$.

(4) Explain why $\mathrm{rank}(AB) \leq \mathrm{rank}(B)$.

(5) Is $\mathrm{rank}(AB) = \min\{\mathrm{rank}(A), \mathrm{rank}(B)\}$?

J*. SOMETHING ELSE ABOUT BASES.

(1) A set of vectors S in a vector space V is a *maximal linearly independent set* if S is linearly independent, and $S \cup \{\mathbf{v}\}$ is linearly dependent for all $\mathbf{v} \notin V$. Explain why a maximal linearly independent set in V is a basis for V.

(2) A set of vectors S in a vector space V is a *minimal spanning set* if S is spans V, and $S \setminus \{\mathbf{s}\}$ does not span V for all $\mathbf{s} \in S$. Explain why a minimal spanning set in V is a basis for V.

[3]Beware of the numbering: if you take the numbers in the same relative places as in the equations above, then A is the transpose of that.