Math 314. Week 7 worksheet (§4.1, §4.2, §4.3).

A **vector space** is a set V with two operations

$$\text{vector addition } + : V \times V \to V \qquad (\text{vector } + \text{ vector } = \text{ vector })$$

$$\text{scalar multiplication } \cdot : \mathbb{R} \times V \to V \qquad (\text{scalar } + \text{ vector } = \text{ vector }),$$

that satisfy a bunch of reasonable conditions. To note a couple:

- we can add any two "vectors" $\mathbf{v}, \mathbf{w} \in V$, and the result $\mathbf{v} + \mathbf{w}$ is always a "vector" in V.
- There is a zero "vector," that we write as $\mathbf{0} \in V$, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.

A **subspace** of a vector space V is a subset $H \subseteq V$ such that

(1) $\mathbf{0} \in H$ — the zero vector of V is an element of H

(2) $\mathbf{v}, \mathbf{w} \in H \Rightarrow \mathbf{v} + \mathbf{w} \in H$ — H is *closed under addition*

(3) $c \in \mathbb{R}, \mathbf{v} \in H \Rightarrow c\mathbf{v} \in H$ — H is *closed under scalar multiplication.*

A subspace of a vector space is a vector space itself!

The **span** of a set of vectors $\{\mathbf{v_1}, \ldots, \mathbf{v_t}\}$ in a vector space V is

$$\begin{aligned}\text{Span}\{\mathbf{v_1}, \ldots, \mathbf{v_t}\} &= \text{set of all linear combinations of } \mathbf{v_1}, \ldots, \mathbf{v_t}\} \\ &= \{c_1\mathbf{v_1} + \cdots + c_t\mathbf{v_t} \mid c_1, \ldots, c_t \in \mathbb{R}\}.\end{aligned}$$

This is always a subspace of V (maybe all of V, maybe smaller).

A. POLYNOMIALS OF DEGREE AT MOST n. Let P_n be the set of polynomials (with variable t) of degree at most n. This is a vector space. The "vectors" are polynomials.

(1) Convince yourself that you can add any two elements of P_n and you always get another element of P_n. Likewise, Convince yourself that you can multiply any element of P_n by a scalar and you always get another element of P_n.

(2) What is the zero "vector" in P_n?

(3) Is the set of $p(t) \in P_n$ such that $p(0) = 1$ a subspace of P_n?

(4) Is the set of $p(t) \in P_n$ such that $p(1) = 0$ a subspace of P_n?

B. MATRICES.

(1) Consider the set of all matrices of all sizes. Explain why this is definitely not a vector space.[1]

(2) The set of all 2×2 matrices is a vector space. What is the zero vector in this vector space?

(3) Describe $\text{Span}\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$.

(4) Explain why the set of diagonal matrices $\begin{bmatrix}\star & 0\\0 & \star\end{bmatrix}$ is a subspace of the vector space of 2×2 matrices.

C. SUBSPACES OF $\mathbb{R}^2$.

(1) Is the upper half plane $\left\{\begin{bmatrix}x\\y\end{bmatrix} \mid x \geq 0\right\}$ a subspace of $\mathbb{R}^2$?

(2) Show that the line $y = 2x$ is equal to $\text{Span}\left\{\begin{bmatrix}2\\1\end{bmatrix}\right\}$. Conclude that this line is a subspace of $\mathbb{R}^2$.

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[1] Hint: Can you add two matrices of different sizes together?

A set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ in a vector space (or subspace) V **spans** V if $\text{Span}\{\mathbf{v_1}, \dots, \mathbf{v_t}\} = V$; equivalently, *every* element of V is a linear combination of $\mathbf{v_1}, \dots, \mathbf{v_t}$.

A set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ in a vector space (or subspace) V is **linearly independent** if the only solution of
$c_1\mathbf{v_1} + \cdots + c_t\mathbf{v_t} = \mathbf{0}$ is $c_1 = \cdots = c_t = 0$.

A set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ in a vector space (or subspace) V is a **basis** for V if is spans V and it is linearly independent.

D. POLYNOMIALS OF DEGREE AT MOST 3. Let P_2 be the set of polynomials (with variable t) of degree at most 2. This is a vector space. Let $S = \{1, t, t^2\}$. This is a set of three particular polynomials.

(1) Show that S spans P_2. This means: show that[2] any polynomial in P_2 is a linear combination of $1, t, t^2$.
(2) Show that S is linearly independent. This means: show that[2] if $c_1 \cdot 1 + c_2 \cdot t + c_3 \cdot t^2$ equals the zero polynomial, then $c_1 = c_2 = c_3 = 0$.
(3) Conclude that S is a basis for P_2.

E. 2×2 MATRICES.

(1) Show that $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for the vector space of 2×2 matrices.
(2) Find a basis for the subspace consisting of diagonal matrices.

F. BASES OF $\mathbb{R}^3$.

(1) Convince yourself that $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ is a basis[3] of $\mathbb{R}^3$.
(2) Consider the set $\left\{ \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix} \right\}$. Notice that when we put them together in a matrix, that matrix is in echelon form. Why does this set span $\mathbb{R}^3$? Why is this set linearly independent? Conclude it is another basis for $\mathbb{R}^3$.
(3) The matrix $A = \begin{bmatrix} -1 & 2 & 6 \\ 4 & -2 & 7 \\ 3 & 4 & -3 \end{bmatrix}$ has RREF $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Does the set of columns of A form a basis of $\mathbb{R}^3$?

If A is an $m \times n$ matrix,

- The **null space of** A is the solution set of $A\mathbf{x} = \mathbf{0}$. It is a subspace of $\mathbb{R}^n$.
- The **column space of** A is the span of the columns of A; equivalently, the set of all vectors $\mathbf{b}$ of the form $\mathbf{b} = A\mathbf{x}$ for all possible $\mathbf{x}$. It is a subspace of $\mathbb{R}^m$.

To find a basis for $\text{Null}(A)$, solve $A\mathbf{x} = \mathbf{0}$, write it in parametric vector form, and take the set of vectors that you use as a basis.

To find a basis for $\text{Col}(A)$, row reduce A to determine which columns are pivot columns; take the set of columns in A that are pivot columns as a basis.

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[2]Don't overthink this. This is quick.
[3]It is called the **standard basis** of $\mathbb{R}^3$.

G. BASES FOR A NULL SPACE AND A COLUMN SPACE.

Let $A = \begin{bmatrix} 2 & 3 & 2 & 1 & 2 \\ -1 & 1 & 4 & 0 & 2 \\ 1 & 0 & -2 & 3 & -4 \end{bmatrix}$. The RREF of A is $\begin{bmatrix} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$.

(1) Find the general solution to the equation $A\mathbf{x} = \mathbf{0}$.
(2) Write your solution from the previous part as a vector in $\mathbb{R}^5$ (where the entries involve the free variables x_3 and x_5), and then write that vector as $x_3\mathbf{v} + x_5\mathbf{w}$ (where $\mathbf{v}$ and $\mathbf{w}$ just have numbers). A basis for $\text{Null}(A)$ is $\{\mathbf{v}, \mathbf{w}\}$.
(3) Which columns of A are pivot columns? Put them in a set; this is a basis for $\text{Col}(A)$.

H*. SUBSPACES OF P_3. Consider the vector space P_4 of polynomials of degree at most three.

(1) Let H be the set of polynomials that have both -1 and 1 as roots. Show that H is a subspace of P_3.
(2) Find a basis for H.

I*. SUBSPACES.

(1) Show that if H and K are two subspaces of V, then $H \cap K$ is a subspace of V.
(2) If H and K are two subspaces of V, is $H \cup K$ a subspace of V?
(3) Show that if H and K are two subspaces of V, then $H + K = \{h + k \mid h \in H, k \in K\}$ is a subspace of V.
(4) If H is a subspace of $\mathbb{R}^n$, S is any basis for H, and T is any basis for $\mathbb{R}^n$, is $S \subseteq T$ always?
(5) If H is a subspace of $\mathbb{R}^n$, and S is any basis for H, can you find a basis T for $\mathbb{R}^n$ that contains S?

J*. SUBSPACES OF $M_{2\times2}$. Consider the vector space $M_{2\times2}$ of 2×2 matrices.

(1) Which of the following are subspaces of $M_{2\times2}$:
  (a) The set of matrices with $A = A^T$.
  (b) The set of all singular matrices A.
  (c) The set of matrices A with $A^2 = 0$.
  (d) The set of matrices with $A = -A^T$.
(2) For each of the above that are subspaces, find a basis.