Worksheet 6 Solutions

A. 1) F   2) T   3) T   4) F

5) T   6) F   7) F   8) T   9) T

B.

1) $\left[\begin{array}{ccc|c} 1 & & & 2 \\ 2 & 1 & & 0 \\ & -1 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & & & 2 \\ & 1 & & -4 \\ & & 1 & -2 \end{array}\right]$ $y = \begin{bmatrix} 2 \\ -4 \\ -2 \end{bmatrix}$

2) $\left[\begin{array}{cccc|c} 3 & 4 & 0 & 1 & 2 \\ & 2 & -8 & 0 & -4 \\ & & 4 & 1 & -2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 6 \\ 0 & 1 & 0 & 1 & -4 \\ 0 & 0 & 1 & 1/4 & -1/2 \end{array}\right]$

$\underline{x} = \begin{bmatrix} 6 \\ -4 \\ -1/2 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ -1/4 \\ 1 \end{bmatrix}$

3) same

4) Agree

C. 1) -12 ; 0

2) -60

3) $2 \det\begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} - (-1)\det\begin{pmatrix} -2 & 2 \\ 4 & 1 \end{pmatrix} + (-1)\det\begin{pmatrix} -2 & -1 \\ 4 & 2 \end{pmatrix}$

$= -(2)\det\begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix} + (-1)\det\begin{pmatrix} 2 & -1 \\ 4 & 1 \end{pmatrix} - \det\begin{pmatrix} 2 & -1 \\ 4 & 2 \end{pmatrix}$

$= -24$

4) $\begin{bmatrix} 2 & -1 & -1 \\ -2 & -1 & 2 \\ 4 & 2 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 5 \end{bmatrix}$

$\det = -20$

D. 1) 5

2) Yes; $1/7$

3) Yes; exactly one

4) 49

5) 7; -7; 21

6) 0

7) 7

8) $2^5 \cdot 7$

9) -7

E 1) Row reduction (just replacements)

$$\rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

$\Rightarrow \det = 24$

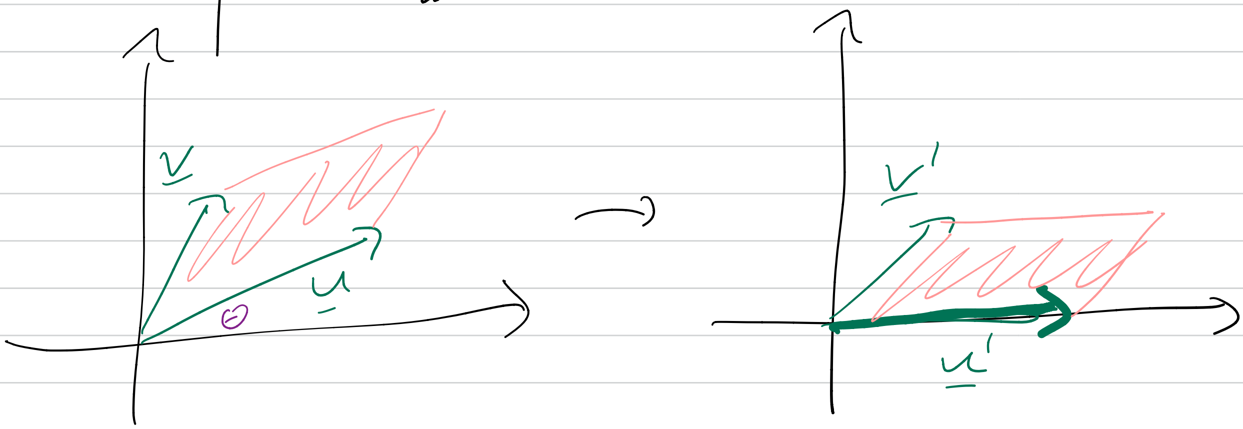
2) $(t+1)(t+2)(t+3) - 3(t+1)$
$\quad - 1(3 - 2(t+2))$
$= t^3 + 6t^2 + 10t + 4.$

Row reduction is bad b/c you have to keep separating cases based on whether certain expressions in t are zero or not.

3) whenever $t^3 + 6t^2 + 10t + 4 \neq 0$.

F. 1) Many possible answers.

Here's one that avoids the worst computations:



Rotate by θ to move $\underline{u}$ to the x-axis.

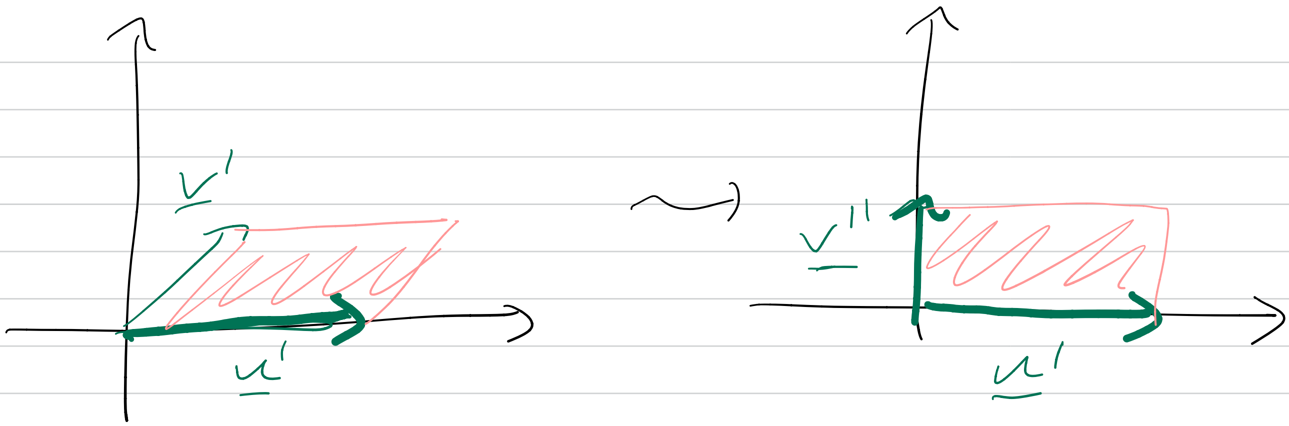
The area stays the same, and

$$\begin{bmatrix} \underline{u}' & \underline{v}' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \underline{u} & \underline{v} \end{bmatrix},$$

so

$$\det\begin{bmatrix} \underline{u}' & \underline{v}' \end{bmatrix} = \underbrace{\boxed{\det\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}}_{= \cos^2\theta + \sin^2\theta = 1} \cdot \det\begin{bmatrix} \underline{u} & \underline{v} \end{bmatrix}$$

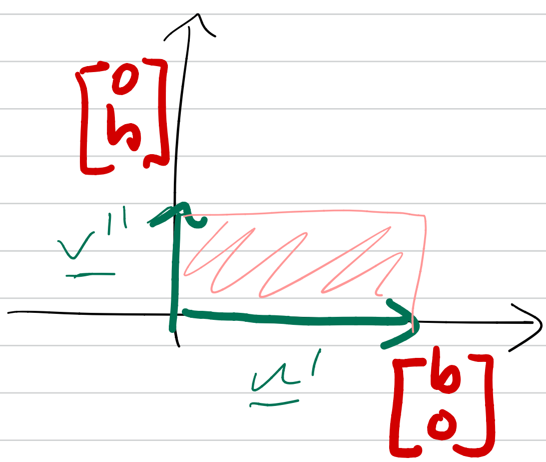
$$= \det\begin{bmatrix} \underline{u} & \underline{v} \end{bmatrix}.$$



Now subtract a multiple of $\underline{u}'$ from $\underline{v}'$ to move $\underline{v}'$ to the y-axis.

Area stays the same (base & height are same), and

$$\det[\underline{u}' \ \underline{v}''] = \det[\underline{u}' \ \ \underline{v}' + t\underline{u}'] = \det[\underline{u}' \ \underline{v}'].$$



Finally, $\det[\underline{u}' \ \underline{v}''] = b \cdot h =$ area.

Altogether, $\det[\underline{u} \ \underline{v}] = \det[\underline{u}' \ \underline{v}''] =$ area.

2. omitted

6.1) there's a free variable.

2) No: If $A\underline{x} = \underline{0}$, then
$\underline{x} = BA\underline{x} = \underline{0}$, so there's only one solution.

3) If A is $m \times n$ with $m < n$, we know there's no B with $BA = I_n$.
Switching roles, if $n < m$ (think of A as "B" in previous part), can't have a B with $AB = I_m$.

4) Many answers, e.g.,

$$\underset{A}{\begin{bmatrix} 1 & 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \underset{I_1}{\begin{bmatrix} 1 \end{bmatrix}}.$$

5) A has a left inverse

$\Longleftrightarrow$ there is a pivot in every column.

A has a right inverse

$\Longleftrightarrow$ there is a pivot in every row.

Explain why!

6) challenge.