# Math 314. Week 6 worksheet (§2.5, §3.1, §3.2).

A. REVIEW. Decide if the following statements are TRUE or FALSE.

(1) All square matrices are invertible.
(2) All invertible matrices are square.
(3) If I know that one column of a (square) matrix is a linear combination of the others, then I decide whether the matrix is invertible or not.
(4) If $\{\mathbf{v_1}, \dots, \mathbf{v_n}\}$ is a set of vectors, and none of them is a scalar multiple of another one of them, then the set is linearly independent.
(5) If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ are vectors in $\mathbb{R}^3$, then $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is definitely linearly dependent.
(6) If $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ are vectors in $\mathbb{R}^3$, then $\operatorname{Span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ is definitely all of $\mathbb{R}^3$.
(7) If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, then $T(\mathbf{x}) = A\mathbf{x}$ for some $3 \times 2$ matrix $A$.
(8) If $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation, then to find its standard matrix, I just need to compute $T(\mathbf{e_1}), T(\mathbf{e_2})$, and $T(\mathbf{e_3})$.
(9) If $T(\mathbf{x})$ is one-to-one, then its standard matrix has a pivot in every column.

B. LU FACTORIZATION. Consider the following equality of matrices $A = LU$:

$$\begin{bmatrix} 3 & 4 & 0 & 1 \\ 6 & 10 & -8 & 2 \\ 0 & -2 & 12 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 0 & 1 \\ 0 & 2 & -8 & 0 \\ 0 & 0 & 4 & 1 \end{bmatrix}.$$

Let $\mathbf{b} = [2, 0, 2]^T$.

(1) Solve the system $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.
(2) Solve the system $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, where $\mathbf{y}$ is the vector you found in part (1).
(3) Use the previous parts[1] to find the solution to $A\mathbf{x} = \mathbf{b}$.
(4) Agree or disagree: This was faster than just solving $A\mathbf{x} = \mathbf{b}$.

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DEFINITION: The **determinant**[2] of a $2 \times 2$ matrix is given by the formula:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

The **determinant** of an $n \times n$ matrix is defined recursively by cofactor expansions:[3]

$$\det A = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j}) = a_{11} \det(A_{11}) - a_{12} \det(A_{12}) + - \cdots + (-1)^{n+1} a_{1n} \det(A_{1n}).$$

You get the same result by taking a cofactor expansion along any row or any column:

$$\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) = (-1)^{i+1} a_{i1} \det(A_{11}) + (-1)^{i+2} a_{12} \det(A_{12}) + \cdots + (-1)^{i+n} a_{1n} \det(A_{in})$$

$$= \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}) = (-1)^{1+j} a_{1j} \det(A_{1j}) + (-1)^{2+j} a_{2j} \det(A_{2j}) + \cdots + (-1)^{n+j} a_{nj} \det(A_{nj}).$$

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[1] This means do not compute anything else!

[2] The **determinant** of a $1 \times 1$ matrix is given by the formula $\det[a] = a$.

[3] In all of these formulas, $A_{ij}$ is the $(n-1) \times (n-1)$ matrix you get by removing row $i$ and column $j$ from $A$.

C. DETERMINANTS.

(1) Compute $\det\begin{bmatrix} 2 & -1 \\ -6 & -3 \end{bmatrix}$ and $\det\begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$.

(2) Compute $\det\begin{bmatrix} 5 & -6 & 1 \\ 0 & -1 & 7 \\ 0 & 0 & 12 \end{bmatrix}$.[4]

(3) Compute $\det\begin{bmatrix} 2 & -1 & -1 \\ -2 & -1 & 2 \\ 4 & 2 & 1 \end{bmatrix}$ by cofactor expansion along the first row. Now compute it by cofactor expansion along the second row.

(4) Use the "row replacement" operation to transform the matrix in the previous part into an echelon matrix, and compute the determinant this way.

D. PROPERTIES OF DETERMINANTS. Suppose that $A$ is a $5 \times 5$ matrix and $\det(A) = 7$. We will write $\mathbf{r_1}, \ldots, \mathbf{r_5}$ for the rows of $A$, and $\mathbf{c_1}, \ldots, \mathbf{c_5}$ for the columns of $A$:

$$A = \begin{bmatrix} \mathbf{r_1} \\ \mathbf{r_2} \\ \mathbf{r_3} \\ \mathbf{r_4} \\ \mathbf{r_5} \end{bmatrix} = \begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} & \mathbf{c_3} & \mathbf{c_4} & \mathbf{c_5} \end{bmatrix}.$$

(1) How many pivots does $A$ have (when we reduce it to echelon form)?

(2) Is $A$ invertible? If so, what is $\det(A^{-1})$?

(3) Does $A\mathbf{x} = \mathbf{b}$ have a solution, where $\mathbf{b} = [2, -3, 5, 4, 1]^T$? If so, how many solutions?

(4) What is $\det(A^2)$?

(5) Compute $\det\left(\begin{bmatrix} \mathbf{r_1} \\ \mathbf{r_2} \\ \mathbf{r_3} - 6\mathbf{r_1} \\ \mathbf{r_4} \\ \mathbf{r_5} \end{bmatrix}\right)$, $\det\left(\begin{bmatrix} \mathbf{r_1} \\ \mathbf{r_4} \\ \mathbf{r_3} \\ \mathbf{r_2} \\ \mathbf{r_5} \end{bmatrix}\right)$, and $\det\left(\begin{bmatrix} \mathbf{r_1} \\ \mathbf{r_2} \\ \mathbf{r_3} \\ 3\mathbf{r_4} \\ \mathbf{r_5} \end{bmatrix}\right)$.

(6) Compute $\det\left(\begin{bmatrix} \mathbf{r_1} \\ \mathbf{r_2} \\ \mathbf{r_2} \\ \mathbf{r_4} \\ \mathbf{r_5} \end{bmatrix}\right)$.

(7) Compute $\det(A^T)$.

(8) Compute $\det(A + A)$.

(9) Compute $\det(\begin{bmatrix} \mathbf{c_1} & \mathbf{c_2} + 3\mathbf{c_4} & \mathbf{c_5} & \mathbf{c_4} & \mathbf{c_3} \end{bmatrix})$.

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[4] Hint: Use a theorem from Section 3.1 on determinants of triangular matrices.

## E. DETERMINANTS

(1) Use the most efficient method to compute $\det \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & 3 & 6 & 6 \\ -1 & 1 & -2 & 2 \end{bmatrix}$.

(2) Compute $\det \begin{bmatrix} t+1 & 1 & 2 \\ 0 & t+2 & 3 \\ -1 & 1 & t+3 \end{bmatrix}$. Your answer will depend on $t$. Is row reduction a good method for computing the determinant here?

(3) For the matrix in the previous part, for which values of $t$ is the matrix invertible?

## F*. SOME PROPERTIES OF DETERMINANTS

(1) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$. Show that the area of the parallelogram with vertices $\mathbf{0}, \mathbf{u}, \mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ is equal to $\det \begin{bmatrix} \mathbf{u} & \mathbf{v} \end{bmatrix}$.

(2) Show that if $A$, $B$ are $2 \times 2$ matrices, then $\det(AB) = \det(A)\det(B)$.

## G*. INVERTING MATRICES.

(1) If $A$ is an $m \times n$ matrix, and $m < n$, explain why the solution set of $A\mathbf{x} = \mathbf{0}$ is infinite.

(2) If $A$ is an $m \times n$ matrix, and $m < n$, can there be a matrix $B$ with $BA = I_n$?

(3) If $A$ is a matrix, and $A$ is not square, can $A$ have an inverse?[5]

(4) Find an $m \times n$ matrix $A$, with $m < n$, and a matrix $B$ such that $AB = I_m$. (We might say that $B$ is the *right inverse of* $A$, and $A$ is the *left inverse of* $B$.)

(5) Characterize which matrices have a left inverse, and which matrices have a right inverse.[6]

(6) Linear transformations are very special; most functions are terrible. Can there be an invertible function $\mathbb{R}^1 \to \mathbb{R}^2$? What about $\mathbb{R}^m \to \mathbb{R}^n$ for any $m, n$?

[5] I.e., can there me a matrix $B$ such that $AB = I_m$ and $BA = I_n$?

[6] Hint: You might express your answers in terms of pivots.