Week 5 worksheet solutions

A. 1. T

2. F

3. F

4. T

5. T

6. T

7. T

8. T

9. F

10. F

B. span; range

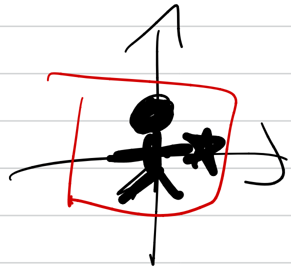
C. 1. yes; 3×3; $\begin{bmatrix} 0 & -1 & 2 \\ -1 & -3 & 6 \\ -3 & -6 & 12 \end{bmatrix}$

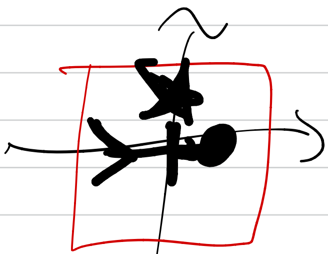
2. $T: \mathbb{R}^2 \to \mathbb{R}^3$, $U: \mathbb{R}^3 \to \mathbb{R}^2$

3. $(T\circ U): \mathbb{R}^3 \to \mathbb{R}^3$

4. $(T\circ U)(\underline{x}) = T(U(\underline{x})) = T(B\underline{x}) = AB\underline{x}$

This means AB is the std. matrix of $(T\circ U)$.

D. 1. Using the picture 

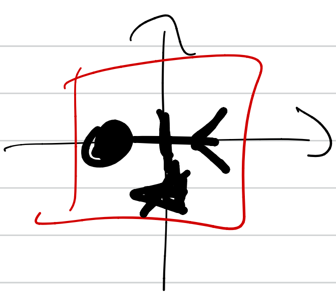
To $\rightsquigarrow$ 

can think of as reflection over line $y=x$.

2. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

3. $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

4. $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

5.  reflection over $y=-x$

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

6. $BA = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

E.1. $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

2. stretches $\longleftrightarrow$ by a & $\updownarrow$ by b.

Undo by stretching $\longleftrightarrow$ by $1/a$ & $\updownarrow$ by $1/b$.

Can only do this when $a, b \neq 0$; inverse is $\begin{bmatrix} 1/a & 0 \\ 0 & 1/b \end{bmatrix}$

3. Stretches by a in x-direction,
by b in y-direction,
by c in z-direction.
Invertible $\Leftrightarrow$ a, b, c all $\neq 0$.

Inverse is $\begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$.

F. 1. $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

2. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. not invertible

4. $\frac{1}{11}\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$.

G. $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 2 & -1 \\ -1 & 2 & -2 \end{bmatrix}$

H. 1) yes (4 pivots)

2) no (3 pivots)

3) no: not square

4) no: columns not lin indep't.

I. 1) $(A^{-1})^T A^T = (A(A^{-1}))^T = I^T = I$

$A^T (A^{-1})^T = (A^{-1} \cdot A)^T = I^T = I$

2) By 1, if A invertible, then A^T has $(A^{-1})^T$ as an inverse.

If A^T is invertible, then $A = (A^T)^T$ is invertible too!

3) $\Leftarrow$ A, B, C are all invertible:

$$(ABC)(C^{-1}B^{-1}A^{-1})$$
$$= ABIB^{-1}A^{-1}$$
$$= ABB^{-1}A^{-1}$$
$$= AIA^{-1}$$
$$= AA^{-1} = I.$$

Similarly for $(C^{-1}B^{-1}A^{-1})(ABC)$.

J.1 If $T(\underline{x}) = \underline{y}$ has unique solution for every $\underline{y}$, then define U to be

$$U(\underline{y}) = (\text{solution to } T(\underline{x}) = \underline{y}).$$

This is a function (since solution is unique) and is defined on all of $\mathbb{R}^4$ (since each $\underline{y}$ has a solution)

Then $(T \circ U)(\cancel{y}) = \underline{x}$

and $(\cancel{U} \circ T)(\underline{x}) = \underline{x}$ for every $\underline{x}$.

So U is the inverse function of T.

If T has an inverse function T^{-1}, then

$$T(\underline{x}) = \underline{y} \implies \underline{x} = T^{-1}(T(\underline{x})) = T^{-1}(\underline{y})$$

exists & is unique since T^{-1} is a function.

2. If $\underline{u}, \underline{v} \in \mathbb{R}^n$, write

$\underline{u} = T(\underline{x})$, $\underline{v} = T(\underline{y})$ for some $\underline{x}, \underline{y} \in \mathbb{R}^n$.

Then
$$\begin{aligned} T^{-1}(\underline{u}+\underline{v}) &= T^{-1}(T(\underline{x}) + T(\underline{y})) \\ &= T^{-1}(T(\underline{x}+\underline{y})) \\ &= \underline{x}+\underline{y} = T^{-1}(\underline{u}) + T^{-1}(\underline{v}). \end{aligned}$$

Likewise,
$$\begin{aligned} T^{-1}(c\underline{u}) &= T^{-1}(cT(\underline{x})) = T^{-1}(T(c\underline{x})) \\ &= c\underline{x} = cT^{-1}(\underline{u}). \end{aligned}$$

3. $T: \mathbb{R}^n \to \mathbb{R}^m$ linear transformation can only be onto if $n \geq m$, and can only be one-to-one if $n \leq m$. By part (1), we must have $m = n$.

4. Yes! Left as a challenge.