Math 314. Week 4 worksheet (§1.8, §1.9, §2.1).

DEFINITION: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$,

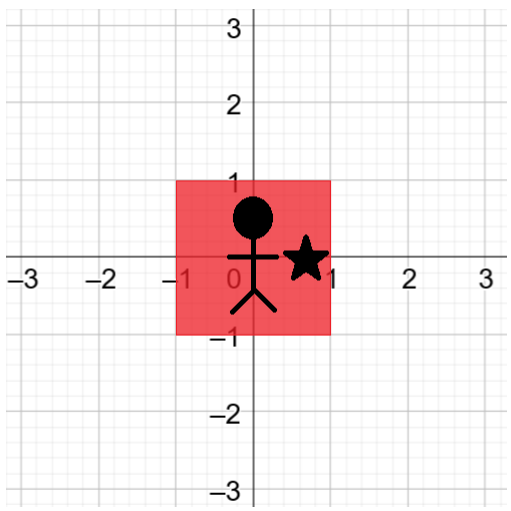
(1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and
(2) $T(c\mathbf{u}) = cT(\mathbf{u})$.

If T is a linear transformation, then

$$T(c_1\mathbf{v_1} + \cdots + c_t\mathbf{v_t}) = c_1T(\mathbf{v_1}) + \cdots + c_tT(\mathbf{v_t}) \quad \text{for all } \mathbf{v_1}, \dots, \mathbf{v_t} \in \mathbb{R}^n, \text{and all } c_1, \dots, c_t \in \mathbb{R}.$$

NOTATION: $\mathbf{e_i}$ denotes the vector with 1 in position i and 0 in every other position. It is sometimes called the **i-th standard vector**.

THEOREM: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} T(\mathbf{e_1}) & \cdots & T(\mathbf{e_n}) \end{bmatrix}$. This matrix A is called the **standard matrix** of T.



A. FAMILIAR LINEAR TRANSFORMATIONS IN $\mathbb{R}^2$. For each of the following linear transformations, draw the image of the picture above under the transformation, and find the standard matrix.

(1) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by "rotate 90 degrees clockwise around the origin."
(2) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by "stretch in the vertical direction by a factor of 2."
(3) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by "shrink in the horizontal direction by a factor of 2."
(4) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by "reflect over the x-axis."
(5) The map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by "project onto the x-axis."

B. MORE MATRIX TRANSFORMATIONS. For each[1] of the following matrices A, consider the linear transformation $T(\mathbf{x}) = A\mathbf{x}$: draw $T(\mathbf{e_1})$, $T(\mathbf{e_2})$, and the image of the picture above under T.

(1) $A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$.

(2) $A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

(3) $A = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$.

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[1]The transformation in part (1) is called a horizontal shear; the transformation in part (2) is called a vertical shear.

DEFINITION: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a function.

- The **domain** of T is $\mathbb{R}^n$.
- The **codomain** of T is $\mathbb{R}^m$.
- The **range** of T is the set of all $T(\mathbf{p})$ for all input values $\mathbf{p}$.
- T is **one-to-one** if $\mathbf{x} \neq \mathbf{y}$ (in the domain) implies $T(\mathbf{x}) \neq T(\mathbf{y})$ (in the codomain).
- T is **onto** if for any $\mathbf{b}$ in the codomain, there is some $\mathbf{x}$ in the domain such that $T(\mathbf{x}) = \mathbf{b}$.

THEOREM: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be its standard matrix. The following are equivalent:

- T is onto.
- $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b} \in \mathbb{R}^m$.
- The columns of A span $\mathbb{R}^m$.
- Any echelon matrix row equivalent to A has a pivot in every row (m pivots).

THEOREM: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be its standard matrix. The following are equivalent:

- T is one-to-one.
- $A\mathbf{x} = \mathbf{b}$ has at most one solution for every $\mathbf{b} \in \mathbb{R}^m$.
- The columns of A are linearly independent.
- Any echelon matrix row equivalent to A has a pivot in every column (n pivots).

C. ROTATION IN $\mathbb{R}^2$, AGAIN. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by "rotate 90 degrees clockwise around the origin."

(1) What are the domain and codomain of T?

(2) Draw the vector $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Can you find some vector $\mathbf{p}$ such that $T(\mathbf{p}) = \mathbf{b}$?

(3) Can any point $\mathbf{b} \in \mathbb{R}^2$ be written as $T(\mathbf{p})$ for some $\mathbf{p}$? What does that say about T in terms of the definitions above?

(4) If $\mathbf{x} \neq \mathbf{y}$, can $T(\mathbf{x}) = T(\mathbf{y})$? What does that say about T in terms of the definitions above?

D. PROJECTION ONTO x-AXIS, AGAIN. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by "project onto the x-axis."

(1) What are the domain and codomain of T?

(2) Find a vector $\mathbf{b}$ that is not in the range of T. Is T onto?

(3) Find two vectors $\mathbf{p} \neq \mathbf{q}$ such that $T(\mathbf{p}) = T(\mathbf{q})$. Is T one-to-one?

E. ANOTHER LINEAR TRANSFORMATION. Consider the linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^4$ with standard matrix

$$A = \begin{bmatrix} 1 & -1 & 3 & -4 & 5 \\ 0 & 2 & 1 & 0 & 3 \\ 5 & -5 & 15 & -19 & 30 \\ 2 & 0 & 7 & -9 & 14 \end{bmatrix}.$$

(1) What are the domain and codomain of T?

(2) Is T one-to-one? Is T onto?

(3) If $\mathbf{b} \in \mathbb{R}^4$, but you don't know which vector, what can you say about the solution set of $A\mathbf{x} = \mathbf{b}$?

DEFINITION: The product of the matrices A and $B = \begin{bmatrix} \mathbf{b_1} & \cdots & \mathbf{b_n} \end{bmatrix}$, is $AB = \begin{bmatrix} A\mathbf{b_1} & \cdots & A\mathbf{b_n} \end{bmatrix}$, whenever $A\mathbf{b_1}, \dots, A\mathbf{b_n}$ are valid products. Otherwise, we cannot take the product AB.

F. MATRIX MULTIPLICATION. Which of the following products AB are possible? If possible, what is the size of the resulting matrix?

(1) A is 2×5 and B is 2×5.
(2) A is 2×5 and B is 5×2.
(3) A is 5×2 and B is 2×5.

G. TRANSFORMATIONS IN $\mathbb{R}^2$. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation "rotate 90 degrees clockwise," and $U : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation "reflect over the x-axis."

(1) Draw the image of the picture from the first page under the composition $T \circ U$. Can you describe the transformation $T \circ U$?[2]
(2) Based on your description of $T \circ U$ from the previous part, compute its standard matrix.
(3) Compute the standard matrix of T and of U. Call then A and B respectively.[3]
(4) Compute AB. Compare to part (2).
(5) Now draw the image of the picture from the first page under the composition $U \circ T$. Describe this map as a single reflection, and find its standard matrix.
(6) Compute BA.

H. MULTIPLICATION AND COMPOSITION. Let T a linear transformation with standard matrix A, and U be a linear transformation with standard matrix $B = \begin{bmatrix} \mathbf{b_1} & \cdots & \mathbf{b_n} \end{bmatrix}$.

(1) Explain why $U(\mathbf{e_1}) = \mathbf{b_1}$.
(2) Explain why $(T \circ U)(\mathbf{e_1}) = A\mathbf{b_1}$.
(3) Use the last part to compute the standard matrix of $T \circ U$.

I. ROTATIONS IN $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation "rotate 90 degrees around the x-axis," and $U : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation "rotate 90 degrees around the y-axis."[4]

(1) Find the standard matrix A for T, and the standard matrix B for U.
(2) Compute AB. What happens when you "rotate 90 degrees around the y-axis" then "rotate 90 degrees around the x-axis?"

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[2]Hint: You can consider it as a single reflection.

[3]You already did this in problem A.

[4]To be precise, rotate counterclockwise if you are looking down from the positive direction on the axis.