Math 314. Week 3 worksheet (§1.5 – §1.8).

A. PARAMETRIC VECTOR FORM. The general solution to the linear system $(\star)$ below is given next to it.

$$(\star)\qquad \begin{aligned} -2x_1 + 3x_2 + 5x_4 - 6x_5 &= -13 \\ -2x_1 - 6x_2 - 4x_4 - 6x_5 + 5x_6 &= 14 \\ 4x_1 + 15x_2 + 11x_4 + 12x_5 - 10x_6 &= -37 \end{aligned}$$

$$\begin{aligned} x_1 &= x_4 - 3x_5 + 2 \\ x_2 &= -x_4 - 3 \\ x_6 &= 0 \\ & x_3, x_4, x_5 \text{ free} \end{aligned}$$

(1) Use the general solution to plug in x_1 through x_6 into the first set of blanks below. Every entry should involve only constants and free variables.

(2) Fill in the blanks after the third "=". Every entry should involve only constants. Now your solution is in parametric vector form.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \\ \underline{\qquad\qquad} \end{bmatrix} = \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix} + x_3 \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix} + x_4 \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix} + x_5 \begin{bmatrix} \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \\ \underline{\quad} \end{bmatrix}.$$

B. SOLUTION SETS AND HOMOGENEOUS SYSTEMS. The general solution to the linear system $(\star)$ below is given next to it.

$$(\star)\qquad \begin{aligned} 2x_1 - 5x_2 + 7x_3 &= 9 \\ x_1 + 4x_3 &= 3 \end{aligned}$$

$$\begin{aligned} x_1 &= 3 - 4x_3 \\ x_2 &= -3/5 - 1/5x_3 \\ & x_3 \text{ free} \end{aligned}$$

(1) Rewrite this solution set in parametric vector form as $\mathbf{x} = \mathbf{p} + s\mathbf{v}$. What geometric shape is the solution set?

(2) Write your solution set in parametric form with a different "constant vector" $\mathbf{p}$.

(3) Without doing any computation, write the solution set of the system

$$\begin{aligned} 2x_1 - 5x_2 + 7x_3 &= 0 \\ x_1 + 4x_3 &= 0 \end{aligned}$$

in parametric vector form.

(4) Given that $(2, -1, 1)$ is a solution to the linear system

$$\begin{aligned} 2x_1 - 5x_2 + 7x_3 &= -2 \\ x_1 + 4x_3 &= 6 \end{aligned},$$

write the solution set to this system in parametric vector form without doing any computation.

DEFINITION: A set of vectors $S = \{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ is **linearly independent** if, whenever $c_1\mathbf{v_1} + \cdots + c_t\mathbf{v_t} = \mathbf{0}$, we have $c_1 = \cdots = c_t = 0$. Otherwise, the set is **linearly dependent**.[1]

C. LINEAR INDEPENDENCE.

(1) Suppose that $2\mathbf{v_1} - 3\mathbf{v_2} + 4\mathbf{v_3} = \mathbf{0}$. Is $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ linearly independent? Write $\mathbf{v_3}$ as a linear combination of $\mathbf{v_1}$ and $\mathbf{v_2}$.

(2) Rewrite the equation $c_1\mathbf{v_1} + \cdots + c_t\mathbf{v_t} = \mathbf{0}$ as a matrix-times-vector equation. Does this correspond to a homogeneous linear system?

(3) Complete the following statement, and explain why it is true:

$S = \{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ is linearly independent if and only if $\begin{bmatrix}\mathbf{v_1} & \cdots & \mathbf{v_t}\end{bmatrix}\mathbf{x} = \mathbf{0}$ has no ____________ solution.

D. LINEAR INDEPENDENCE. Determine if each of the following sets of vectors is linearly independent.

(1) $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\-4\end{bmatrix}\right\}$.

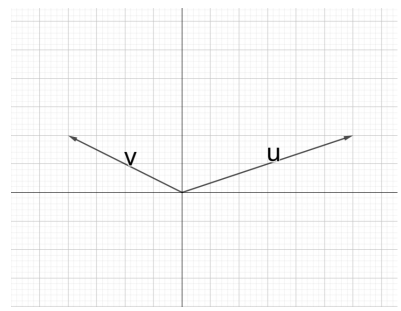
(2) $\left\{\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\5\end{bmatrix}\right\}$.

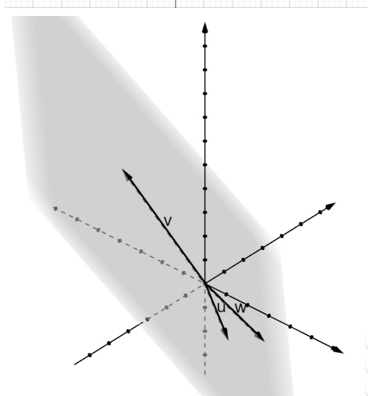
(3) $\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\5\end{bmatrix}\right\}$.

(4) $\left\{\begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}-2\\5\end{bmatrix}\right\}$.

(5) $\left\{\begin{bmatrix}1\\1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\2\\2\\3\end{bmatrix}, \begin{bmatrix}2\\2\\3\\3\end{bmatrix}, \begin{bmatrix}3\\3\\3\\4\end{bmatrix}\right\}$.

(6) $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}\} \subset \mathbb{R}^4$.

(7) 

(8) 

E*. LINEAR INDEPENDENCE AND RREF.

(1) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} \subseteq \mathbb{R}^3$ form a linearly independent set, what are all the possible RREFs for $\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3}\end{bmatrix}$?

(2) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} \subseteq \mathbb{R}^4$ form a linearly independent set, what are all the possible RREFs for $\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3}\end{bmatrix}$?

(3) If $\{\mathbf{v_1}, \mathbf{v_2}, \dots, \mathbf{v_n}\} \subseteq \mathbb{R}^m$ form a linearly independent set, what are all the possible RREFs for $\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_n}\end{bmatrix}$? What can you say about m and n?

(4) If $\{\mathbf{v_1}, \mathbf{v_2}\} \subseteq \mathbb{R}^3$ form a linearly dependent set, what are all the possible RREFs for $\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2}\end{bmatrix}$?[2]

(5) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\} \subseteq \mathbb{R}^3$ form a linearly dependent set, what are all the possible RREFs for $\begin{bmatrix}\mathbf{v_1} & \mathbf{v_2} & \mathbf{v_3}\end{bmatrix}$?

---

[1] Linear (in)dependence is a property of a set of vectors: it's about their relationship with each other. However, people will often say things like "$\mathbf{v_1}, \dots, \mathbf{v_t}$ are linearly independent" to mean "$\{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ is linearly independent."

[2] Hint: Depending how you organize them, you should have 3 or 4 "types" of matrices.

DEFINITION: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all $c \in \mathbb{R}$,

(1) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$, and
(2) $T(c\mathbf{u}) = cT(\mathbf{u})$.

These two conditions are equivalent to the single condition

(†) $$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v}) \quad \text{for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n, \text{and all } c, d \in \mathbb{R}.$$

DEFINITION: A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **matrix transformation** if $T(\mathbf{x}) = A\mathbf{x}$ for some matrix A.

THEOREM: Every matrix transformation is a linear transformation.

F. LINEAR TRANSFORMATIONS.

(1) Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule $T(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}) = (\begin{bmatrix} x_1^2 \\ x_2^2 \end{bmatrix})$. Compare $T(\begin{bmatrix} 2 \\ 0 \end{bmatrix})$ and $2T(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$. Is T a linear transformation?
(2) Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\mathbf{x}) = -\mathbf{x}$. Is T a linear transformation?
(3) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and $T(\begin{bmatrix} 1 \\ 0 \end{bmatrix}) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $T(\begin{bmatrix} 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, then what is $T(\begin{bmatrix} 2 \\ 2 \end{bmatrix})$?

G. MATRIX TRANSFORMATIONS. Describe each of the following matrix transformations geometrically.[3]

(1) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.
(2) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by the rule $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
(3) $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by the rule $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

H*. PROPERTIES OF LINEAR TRANSFORMATIONS. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation.

(1) Decide on a suitable definition of "line" in $\mathbb{R}^n$. Show that if $L \subseteq \mathbb{R}^n$ is a line, then $T(L)$ is either a line or a point[4].
(2) If $\{\mathbf{v_1}, \ldots, \mathbf{v_t}\} \subset \mathbb{R}^n$ is linearly independent, must $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_t})\} \subset \mathbb{R}^m$ be linearly independent?
(3) If $\{T(\mathbf{v_1}), \ldots, T(\mathbf{v_t})\} \subset \mathbb{R}^m$ is linearly independent, must $\{\mathbf{v_1}, \ldots, \mathbf{v_t}\} \subset \mathbb{R}^n$ be linearly independent?
(4) If $m = n = 2$ and $W \subset \mathbb{R}^2$ is a square, what can you say about $T(W)$? What if W is a triangle? A circle?
(5) Suppose that $m = n$, and $U : \mathbb{R}^n \to \mathbb{R}^n$ is an inverse function to T ($U \circ T = T \circ U$ is the identity function on $\mathbb{R}^n$). Show that U is a linear transformation.

---

[3]Hint: Plug in $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ or $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

[4]By definition, $T(L) = \{T(\mathbf{p}) \mid \mathbf{p} \in L\}$.