Week 2 worksheet solutions

A. 1) no; no; yes; yes

2) $3x_2 - 6x_3 + 6x_4 + 4x_5 = 5$

3; 5.

The last is corresponds to the constants; the others correspond to (coefficients of) a variable.

3)
$$\begin{bmatrix} \blacksquare & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \blacksquare & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \blacksquare & \cdot \end{bmatrix}$$

basic: x_1, x_2, x_5

free: x_3, x_4

4) There is no $[0 \cdots 0 \; b]$ $b \neq 0$ row.

5)
$$x_1 = 2x_3 - 3x_4 - 24$$
$$x_2 = 2x_3 - 2x_4 - 7$$
$$x_5 = 4$$
x_3, x_4 free

B. 1) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 3 & -1 & -1 & -2 \\ 2 & -3 & 2 & 14 \end{array}\right]$

3) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & -7 & -4 & 2 \end{array}\right]$ ($R2 - 3R1$, $R3 - 2R1$)

4) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & -5 & -10 & -20 \\ 0 & 0 & 10 & 30 \end{array}\right]$

6) $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$

7) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$ ($R1 - 3R3$ then $R1 - 2R2$, $R2 - 2R3$)

C.1) 7 variables, 4 equations

2) · no solution POSSIBLE

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{A} \qquad \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\underline{b}}$$

many different answers are possible

· exactly one solution IMPOSSIBLE

There are more variables than rows so not every column can have a pivot. Thus, there is a free variable, so if a solution exists, infinitely many do.

· infinitely many POSSIBLE

$$\underset{A}{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\underline{b}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

many different answers are possible

3) 4 variables & 7 equations

4) · no solution POSSIBLE

$$\underset{B}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\underline{b}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

· exactly one POSSIBLE

$$\underset{B}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}} \quad \underset{\underline{b}}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}$$

- infinitely many POSSIBLE

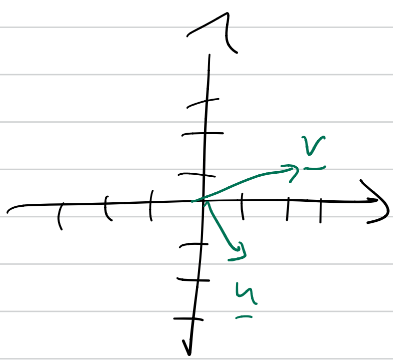
$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$B \qquad \underline{b}$

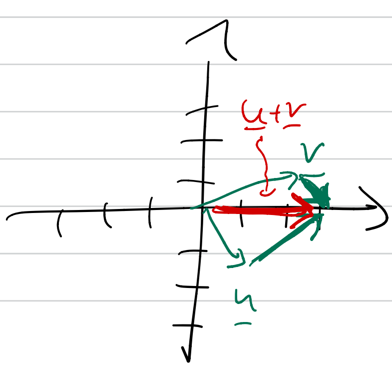
5) For 4×7, ∞ solutions,
for 7×4, no solution.

D. 1) $\begin{bmatrix} 3 \\ 0 \end{bmatrix}$

2)



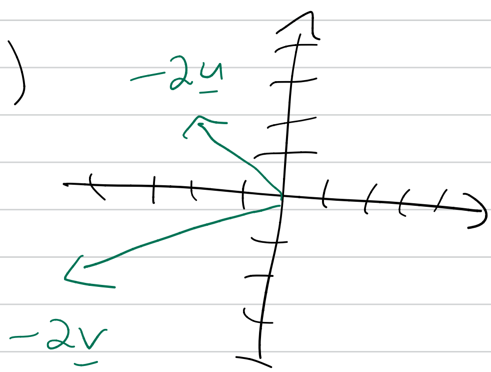
3)



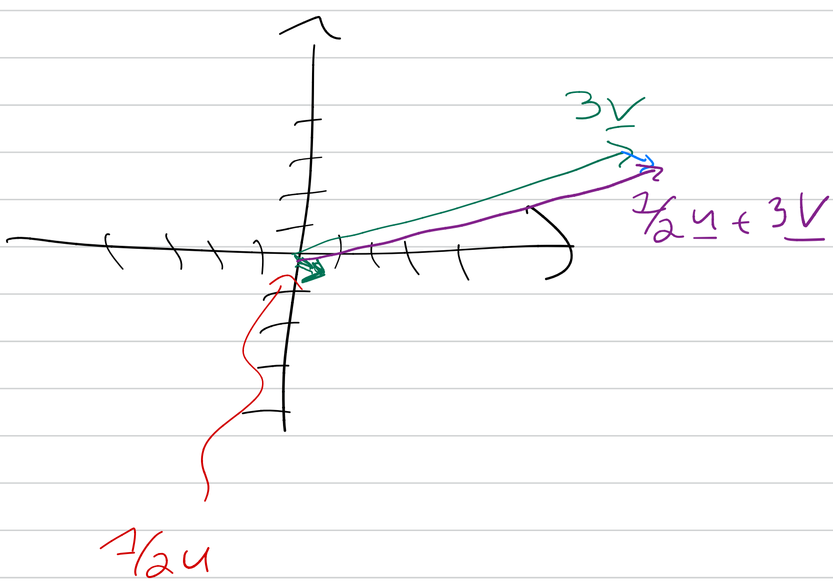
4) $\underline{u}+\underline{v}$ is the fourth corner of the parallelogram with vertices $\underline{0}$, $\underline{u}$, and $\underline{v}$.

5) $\begin{bmatrix} -4 \\ -2 \end{bmatrix}$

6)



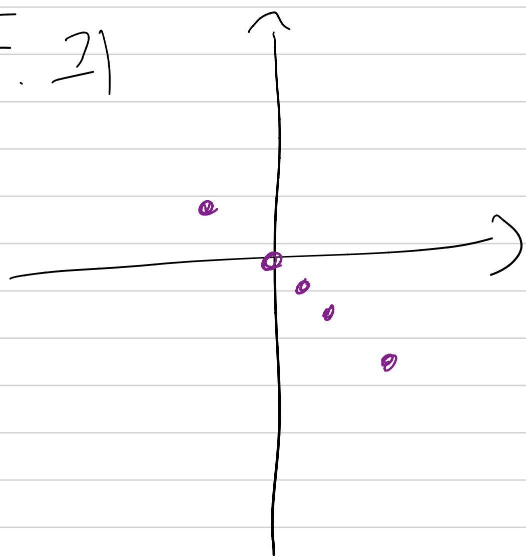
7)



E. 1) $A\underline{b} = 7\begin{bmatrix}1\\4\end{bmatrix} + 8\begin{bmatrix}2\\5\end{bmatrix} + 9\begin{bmatrix}3\\6\end{bmatrix}$.

2) $\begin{bmatrix}3 & 2 & 9\\9 & 7 & 1\end{bmatrix}\begin{bmatrix}5\\13\\-42\end{bmatrix}$.

F. 1)



they are all in Span$\{\underline{u}\}$ by definition

2) $\underline{u} = 1\cdot\underline{u} + 0\cdot\underline{v}$

$\underline{u}+\underline{v} = 1\cdot\underline{u} + 1\cdot\underline{v}$.

For any $\begin{bmatrix}a\\b\end{bmatrix} \in \mathbb{R}^2$, we can solve

$c_1\begin{bmatrix}1\\-1\end{bmatrix} + c_2\begin{bmatrix}2\\1\end{bmatrix} = \begin{bmatrix}a\\b\end{bmatrix}$, since

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ -1 & 1 & b \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & 3 & a+b \end{array}\right]$$ echelon form has no $[0 \cdots 0 \mid b]$ row.

3) For any $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$,

can solve

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}:$$

$x_1 = a$, $x_2 = b$, and $x_3 = c$!

4) Is
$$\begin{cases} x_1 + 4x_2 + 7x_3 = -1 \\ 2x_1 + 5x_2 + 8x_3 = 5 \\ 3x_1 + 6x_2 + 9x_3 = 9 \end{cases}$$
consistent?

6.1) $A\underline{p} = 3\underline{v_1} + (-1)\underline{v_2} + 0\underline{v_3} + \cdots + 0\underline{v_6}$

$= 3\underline{v_1} - \underline{v_2} = 3\underline{v_1} - 3\underline{v_1} = \underline{0}$.

2) $\underline{0} = A\underline{p} = 3\underline{v_1} + (-1)\underline{v_2} + 0\underline{v_3} + \cdots + 0\underline{v_6}$

$= 3\underline{v_1} - \underline{v_2}$

So $\underline{v_2} = 3\underline{v_1}$.

3) Call this matrix "B".

$\underline{0}$ = Column 3 of $B = B \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

So $\underline{0} = A \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{v_3}$.

By (1) (essentially) $B \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \underline{0}$, so $A \cdot \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ -1 \\ 0 \end{bmatrix} = \underline{0}$,

So $\underline{v_5} = 3\underline{v_1}$ by (2) (essentially).

4) In B, column 4 = col 2 − col 1.

This means $\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ is a solution to $B\underline{x} = \underline{0}$,

so also of $A\underline{x} = \underline{0}$, so

$$\underline{v_4} = \underline{v_2} - \underline{v_1}.$$

That's not true, so no!