Math 314. Week 2 worksheet (§1.1 – §1.5).

A. REDUCED ECHELON FORM AND SOLUTIONS. Consider the following two matrices:

$$A = \begin{bmatrix} 0 & 3 & -6 & 6 & 4 & 5 \\ 3 & -7 & 8 & -5 & 8 & 9 \\ 3 & -9 & 12 & -9 & 6 & 15 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & -2 & 3 & 0 & -24 \\ 0 & 1 & -2 & 2 & 0 & -7 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}.$$

I used a sequence of elementary row operations to transform A into B.[1]

(1) Is the matrix A in echelon form? Reduced echelon form? Is B in echelon form? Reduced echelon form?
(2) Suppose that A is the *augmented matrix* of a linear system that we will call "($\star$)". Write down the first equation in the system ($\star$). How many equations are in this system? How many variables? How is the role of the last column of A different from the role of the others in this linear system?
(3) What are the pivots of A (or B)? Which variables in ($\star$) are *basic* variables, and which ones are *free* variables?
(4) Explain how, without computing anything else, we know that the linear system ($\star$) is consistent.
(5) Write down the general solution to the linear system ($\star$).

B. ROW REDUCTION ALGORITHM.

$$\begin{cases} x + 2y + 3z = 6 \\ 3x + y - z = -2 \\ 2x - 3y + 2z = 14 \end{cases}$$

(1) Rewrite this system of equations as an augmented matrix.
(2) Find the first (left-most) nonzero column. If the top entry is 0, use the exchange operation to move a nonzero entry there.[2]
(3) Use the replacement operation (perhaps a few times) to turn every entry below the pivot into 0.
(4) Repeat the first few steps until you run out of rows.
(5) Congratulate yourself: your matrix is in echelon form.
(6) Scale each row to turn every pivot into a 1.
(7) Working from right to left, use the replacement operation to turn each entry above a pivot into 0.
(8) Congratulate yourself: your matrix is in reduced echelon form.
(9) Find the general solution to your system of equations.

C*. MATRIX SHAPES AND SOLUTIONS.

(1) Suppose that A is a 4×7 matrix (4 rows and 7 columns), and that $\mathbf{b}$ is a vector in $\mathbb{R}^4$. Consider the equation $A\mathbf{x} = \mathbf{b}$. If we rewrite $A\mathbf{x} = \mathbf{b}$ as a linear system, how many variables and how many equations will there be?
(2) Without knowing anything else about A and $\mathbf{b}$, determine if each of the following is POSSIBLE or IMPOSSIBLE:
   - $A\mathbf{x} = \mathbf{b}$ has no solution.
   - $A\mathbf{x} = \mathbf{b}$ has exactly one solution.
   - $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

   If "possible" come up with a specific example of an A and $\mathbf{b}$. If "impossible" explain why not.

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[1] Trust me! Don't bother checking it.

[2] It's not! So actually, don't do anything.

(3) Now suppose that B is a 7×4 matrix (7 rows and 4 columns), and that $\mathbf{c}$ is a vector in $\mathbb{R}^7$. Consider the equation $B\mathbf{x} = \mathbf{c}$. If we rewrite $B\mathbf{x} = \mathbf{c}$ as a linear system, how many variables and how many equations will there be?

(4) Without knowing anything else about B and $\mathbf{c}$, determine if each of the following is POSSIBLE or IMPOSSIBLE:

- $B\mathbf{x} = \mathbf{c}$ has no solution.
- $B\mathbf{x} = \mathbf{c}$ has exactly one solution.
- $B\mathbf{x} = \mathbf{c}$ has infinitely many solutions.

If "possible" come up with a specific example of an B and $\mathbf{c}$. If "impossible" explain why not.

(5) In part (2) above, which option do you think is most likely? Same question for part (4).

D. VECTOR OPERATIONS. Let $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$.

(1) Compute $\mathbf{u} + \mathbf{v}$ by adding the coordinates $\mathbf{u}$ and $\mathbf{v}$.

(2) Draw $\mathbf{u}$ and $\mathbf{v}$ as arrows on the plane.[3]

(3) Draw an arrow starting at $(1, -1)$ that goes 2 to the right and 1 up: we think of this as "moving the tail of $\mathbf{v}$ to the head of $\mathbf{u}$." Now draw an arrow starting at $(2, 1)$ that goes 1 to the right and 1 down. Finally, draw $\mathbf{u} + \mathbf{v}$.

(4) State the "parallelogram rule for addition."

(5) Compute $-2\mathbf{v}$ by multiplying each coordinate of $\mathbf{v}$ by -2.

(6) Draw $-2\mathbf{v}$ using the previous part. Now draw $-2\mathbf{u}$ without doing any computation.

(7) Try to draw $\frac{1}{2}\mathbf{u} + 3\mathbf{v}$ just by drawing pictures; no computation.

DEFINITION: Let $\mathbf{v_1}, \dots, \mathbf{v_t}$ be vectors in $\mathbb{R}^n$. A vector of the form $c_1\mathbf{v_1} + \cdots + c_t\mathbf{v_t}$ for some $c_1, \dots, c_t \in \mathbb{R}$ is a *linear combination* of $\mathbf{v_1}, \dots, \mathbf{v_t}$; specifically, it is the linear combination with *weights* $c_1, \dots, c_t$. The *span* of $\mathbf{v_1}, \dots, \mathbf{v_t}$ is the set of all linear combinations of $\mathbf{v_1}, \dots, \mathbf{v_t}$ in $\mathbb{R}^n$.

E. LINEAR COMBINATIONS AND MATRIX EQUATIONS. In this problem, DO NOT multiply or add up any numbers!

(1) Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and the vector $\mathbf{b} = \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$. Express the product $A\mathbf{b}$ explicitly as a linear combination of the columns of A.

(2) Consider the vectors $\mathbf{v_1} = \begin{bmatrix} 3 \\ 9 \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$, and $\mathbf{v_3} = \begin{bmatrix} 9 \\ 1 \end{bmatrix}$. Express the linear combination $5v_1 + 13v_2 - 42v_3$ in the form $A\mathbf{b}$.

[3]Specifically, draw an arrow from $(0, 0)$ to $(1, -1)$ for $\mathbf{u}$.

F. SPAN.

(1) Let $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \in \mathbb{R}^2$. Draw (as points in the plane) $1\mathbf{u}$, $0\mathbf{u}$, $-1\mathbf{u}$, $1/2\mathbf{u}$, and $2\mathbf{u}$. All these points are on a line; is every point on this line in $\mathrm{Span}\{\mathbf{u}\}$?

(2) Let $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in \mathbb{R}^2$. Is $\mathbf{u} \in \mathrm{Span}\{\mathbf{u}, \mathbf{v}\}$? What about $\mathbf{u} + \mathbf{v}$? Convince yourself that $\mathrm{Span}\{\mathbf{u}, \mathbf{v}\} = \mathbb{R}^2$.

(3) Convince yourself that $\mathrm{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ includes every point in $\mathbb{R}^3$.

(4) Rephrase the question

$$\text{Is } \begin{bmatrix} -1 \\ 5 \\ -9 \end{bmatrix} \text{ in the span of } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\} ?$$

in terms of a system of linear equations.

G*. RELATIONSHIP BETWEEN COLUMNS AND SOLUTIONS. Let $A = \begin{bmatrix} \mathbf{v_1} & \mathbf{v_2} & \cdots & \mathbf{v_t} \end{bmatrix}$ be a matrix, where the $\mathbf{v_i}$'s are columns.

(1) Suppose that $\mathbf{v_2} = 3\mathbf{v_1}$. Explain why the vector $\mathbf{p} = \begin{bmatrix} 3 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$.

(2) Conversely, if $\mathbf{x} = \mathbf{p}$ is a solution to $A\mathbf{x} = \mathbf{0}$, explain why $\mathbf{v_2} = 3\mathbf{v_1}$.

(3) Now suppose that the RREF of A is

$$\begin{bmatrix} 1 & 0 & 0 & -1 & 3 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

What can you say about $\mathbf{v_3}$ in the original matrix A? What can you say about the relationship of $\mathbf{v_1}$ and $\mathbf{v_5}$ in A?

(4) With the same info as in part (3), *without doing any row reduction*[4], could A be

$$\begin{bmatrix} 2 & 3 & 0 & -1 & 6 & 7 \\ -6 & 1 & 0 & -7 & -18 & -5 \\ 1 & 5 & 0 & -4 & 3 & 6 \end{bmatrix}?$$

[4] Hint: What is the relationship between $\mathbf{v_1}$, $\mathbf{v_2}$, and $\mathbf{v_4}$?