Math 314. Week 14 worksheet (review).

A. WARMUP TRUE/FALSE.

(1) The domain of the linear transformation with the standard matrix $\begin{bmatrix} 2 & -3 & 1 \\ 0 & 5 & -7 \end{bmatrix}$ is $\mathbb{R}^2$.

(2) If A and B are square matrices, then $\det(A+B) = \det(A) + \det(B)$.

(3) If B is an echelon form for a matrix A, then the pivot columns of B form a basis for $\mathrm{Col}(A)$.

(4) If the columns of an $n \times n$ matrix span $\mathbb{R}^n$, then the determinant of that matrix is nonzero.

(5) A scalar λ is an eigenvalue for A if and only if $A - \lambda I$ is not invertible.

(6) If A is a symmetric matrix, then A is diagonalizable.

(7) If A is diagonalizable, then A is invertible.

(8) If $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is an orthogonal basis for a subspace $W \subset \mathbb{R}^n$, then $\{\mathbf{v_1}, 2\mathbf{v_2}, 3\mathbf{v_3}\}$ is an orthogonal basis for W.

(9) If T is a 4×4 triangular matrix with 4 different entries on the diagonal, then it is diagonalizable.

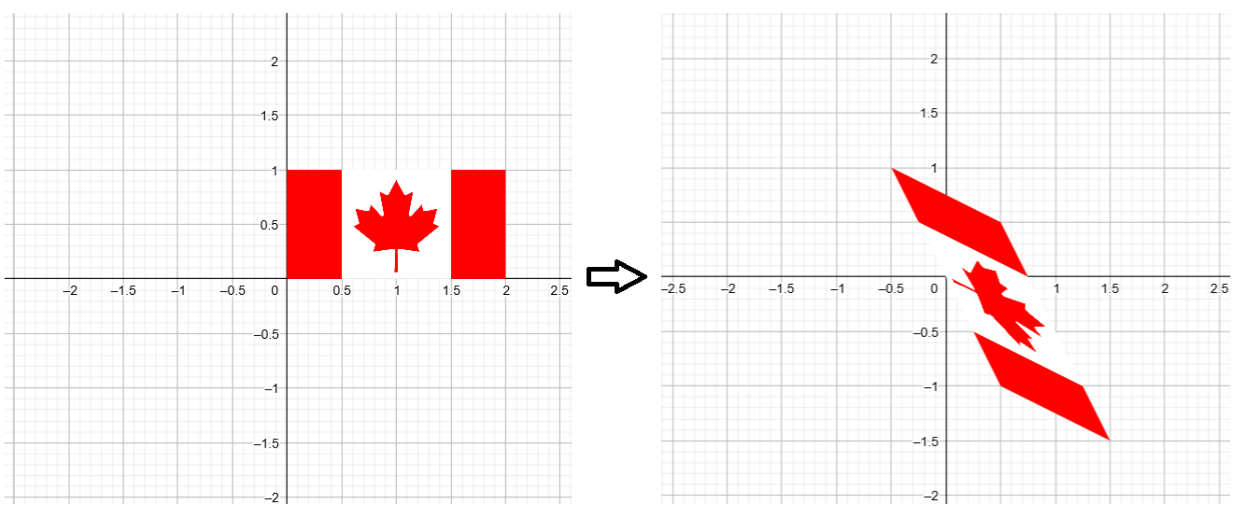
(10) We can compute the eigenvalues of a matrix by row reducing to echelon form and taking the entries on the diagonal.

B. LINEAR TRANSFORMATIONS FROM $\mathbb{R}^2 \to \mathbb{R}^2$.

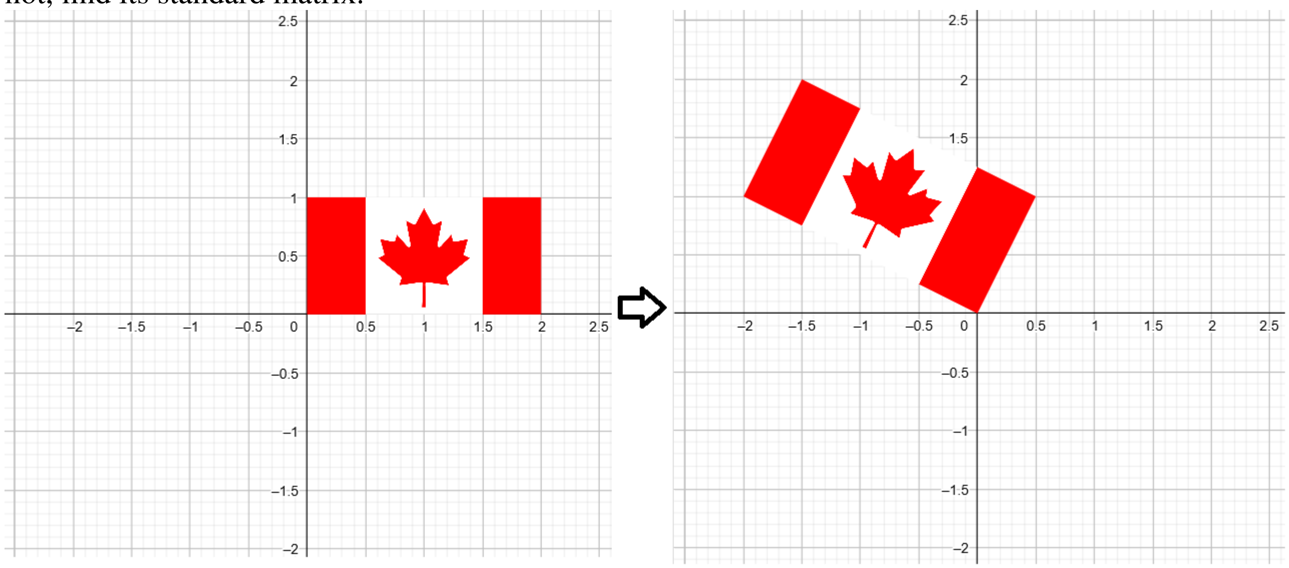
(1) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, how can you find the first column of its standard matrix? The second column?

(2) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1/2 \end{bmatrix}$, compute $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$ and $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. What is the standard matrix of T?

(3) Is the transformation depicted below a linear transformation? Why or why not? If so, find its standard matrix.



(4) Is there any reason why the transformation depicted below isn't a linear transformation? If not, find its standard matrix.



C. Vector spaces

(1) What two operations does a vector space have?
(2) Give at least four different types of examples of vector spaces.
(3) What "special element" does a vector space have? Name it for each of your examples of vector spaces.

D. Subspaces

(1) What three conditions need to be checked to see if a subset is a subspace?
(2) If $\mathbf{v}, \mathbf{w} \in V$, is $\{s\mathbf{v} + t\mathbf{w} \mid s, t \in \mathbb{R}\}$ a subspace of V? Hint: what is another name for this set?
(3) If $\mathbf{v} \in V$ is a nonzero vector, is $\{s\mathbf{v} \mid s \geq 0\}$ a subspace of V? Check the three conditions for a subspace.
(4) How many vectors are in $\operatorname{Span}\left\{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -3 \end{bmatrix}\right\}$?
(5) How many vectors are in $\operatorname{Span}\{\mathbf{0}\}$?

E. Bases and dimension

(1) If V is two-dimensional and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$, is $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ a linearly independent set? Is $\{\mathbf{u}, \mathbf{v}\}$ a linearly independent set? Does $\{\mathbf{u}, \mathbf{v}\}$ span V?
(2) If $V = \operatorname{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, what can you say about $\dim(V)$? How could you find $\dim(V)$?
(3) If $2\mathbf{v} + 3\mathbf{w} = 7\mathbf{u}$, then what are the possible dimensions for $\operatorname{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$?
(4) How many different bases are there for $\mathbb{R}^2$?

F. SOLVING A LINEAR SYSTEM. Consider the augmented matrix $\left[\begin{array}{ccc|c} 3 & -2 & 1 & 7 \\ -6 & 5 & 1 & 2 \\ 0 & 5 & 1 & -8 \end{array}\right]$

(1) Write the linear system associated to this augmented matrix.
(2) Rewrite this linear system as a vector equation.
(3) Rewrite this linear system as a matrix-times-vector equation.
(4) Find the general solution of this system.

G. SOLUTIONS OF LINEAR SYSTEMS

(1) Given a linear system, how can we tell if the system has no solution? one solution? infinitely many solutions?
(2) Can $A\mathbf{x} = \mathbf{b}$ have a solution and $A\mathbf{x} = \mathbf{0}$ not have a solution?
(3) Can $A\mathbf{x} = \mathbf{0}$ have a solution and $A\mathbf{x} = \mathbf{b}$ not have a solution?
(4) Can $A\mathbf{x} = \mathbf{b}$ have exactly one solution $A\mathbf{x} = \mathbf{0}$ not have exactly one solution?
(5) If A is a 4×7 matrix, and $A\mathbf{x} = \mathbf{b}$, what size is $\mathbf{x}$ and what size is $\mathbf{b}$?
(6) Is there a 4×7 matrix A for which $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$? Given A, how would you tell?
(7) Is there a 4×7 matrix A for which $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $\mathbf{b}$? Given A, how would you tell?
(8) Is there a 7×4 matrix A for which $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$? Given A, how would you tell?
(9) Is there a 7×4 matrix A for which $A\mathbf{x} = \mathbf{b}$ has only one solution for every $\mathbf{b}$ for which the system is consistent? Given A, how would you tell?

H. LINEAR TRANSFORMATION. Consider the function $E : P_2 \to \mathbb{R}^4$ given by $E(p(t)) = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \end{bmatrix}$.

(1) What two conditions do we check to see if E is a linear transformation?
(2) Check carefully that E is a linear transformation.
(3) What is the kernel of E?
(4) What is the range of E? Find a basis for it.

I. $\mathcal{B}$-MATRIX. Consider the function $D : P_2 \to P_2$ given by $D(p(t)) = \frac{dp}{dt}$.

(1) The set of polynomials $\mathcal{B} = \{t^2, (t-1)^2, (t-2)^2\}$ is a basis for P_2. So is $\mathcal{C} = \{t^2, t, 1\}$. Find the change-of-coordinates matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$.
(2) Find the $\mathcal{B}$-matrix of D.

J. MORE $\mathcal{B}$-MATRICES

(1) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection over the line L through the origin, find a basis $\mathcal{B}$ for $\mathbb{R}^2$ such that $[T]_{\mathcal{B}} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.
(2) If $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are linearly independent eigenvectors for a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, with eigenvalues $\lambda_1, \ldots, \lambda_n$, what is the $\mathcal{B} = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$-matrix for T? What is the standard matrix for T?
(3) If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation by $\pi/3$ counterclockwise, and $\mathcal{B}$ is an orthonormal basis for $\mathbb{R}^2$, can you determine $[T]_{\mathcal{B}}$?