Math 314. Week 13 worksheet (§7.1, §7.4).

A matrix A is **symmetric** if $A^T = A$. Note that every symmetric matrix is square.

A matrix A is **orthogonal** if it is square, and the columns of A form an orthonormal set.

A. SYMMETRIC MATRICES.

(1) Which of the following 3×3 matrices are symmetric?

- $\begin{bmatrix} 5 & 4 & 1 \\ 6 & 2 & 4 \\ 3 & 6 & 5 \end{bmatrix}$
- $\begin{bmatrix} 6 & 5 & 4 \\ 5 & 3 & 2 \\ 4 & 2 & 1 \end{bmatrix}$
- $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & 3 \\ 5 & 6 & 5 \end{bmatrix}$
- $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 1 & 3 & 5 \end{bmatrix}$

(2) If A is any matrix, is $B = A^T A$ always symmetric? Why or why not?
(3) Is $A = \begin{bmatrix} \mathbf{a_1} & \cdots & \mathbf{a_n} \end{bmatrix}$,explain how the entries of $A^T A$ are related to dot products between the vectors $\mathbf{a_i}$.
(4) If B is a symmetric matrix, is $B = A^T A$ always for some matrix A? Why or why not?[1]

B. ORTHOGONAL MATRICES.

(1) Explain why an $n \times n$ matrix Q is orthogonal if and only if $Q^T Q = I_n$.
(2) Explain why an orthogonal matrix Q must be invertible and $Q^{-1} = Q^T$.
(3) For which of the following linear transformations from $\mathbb{R}^2 \to \mathbb{R}^2$ is its standard matrix orthogonal?
   - rotation counterclockwise by $\alpha°$.
   - stretching by a in the vertical direction.
   - reflection over the y-axis.

SPECTRAL THEOREM: If A is a symmetric matrix, then A is diagonalizable. Moreover, there is an *orthogonal* matrix Q (and diagonal matrix D) such that $A = QDQ^T$.

An expression $A = QDQ^T$ with Q orthogonal and D diagonal is called an **orthogonal diagonalization** of A. One of the key points behind the spectral theorem is the following:

THEOREM: If A is a symmetric matrix, and $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors of A with different eigenvalues, then $\mathbf{v}$ and $\mathbf{w}$ are orthogonal.

C. SPECTRAL THEOREM Which of the following are true?

(1) Every $n \times n$ matrix is similar to a diagonal matrix.
(2) Every $n \times n$ symmetric matrix is similar to a diagonal matrix.
(3) Every $n \times n$ matrix has an eigenvalue.
(4) Every $n \times n$ symmetric matrix has an eigenvalue.
(5) Every $n \times n$ symmetric matrix has n distinct eigenvalues.
(6) Every basis of an eigenspace of an $n \times n$ symmetric matrix is orthonormal.
(7) For every $n \times n$ symmetric matrix, there is an orthonomal basis consisting of eigenvectors.

---

[1]Hint: Use the previous part.

D. Computing an orthogonal diagonalization In this problem, we will find an orthogonal diagonalization of the matrix $B = \begin{bmatrix} 23 & 8 & 8 \\ 8 & 11 & -16 \\ 8 & -16 & 11 \end{bmatrix}$.

(1) First, we find the eigenvalues of B. Explain how to find the eigenvalues of B (but don't go through the computation). Your answer should involve the word "roots."

I computed that the eigenvalues of B are $\lambda = 27$ and $\lambda = -9$.

(2) Second, we find bases for the eigenspaces of B. Explain how to find bases for the eigenspaces of B (but don't go through the computation).

A basis of the $\lambda = 27$ eigenspace is $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$, and a basis for the $\lambda = -9$ eigenspace is $\left\{ \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(3) Third, we apply the Gram-Schmidt process to turn bases for any eigenspaces of dimension > 1 into orthogonal sets. Do this step. Now explain why, once we combine all of the bases for the different eigenspaces together, we get an orthogonal set.

(4) Fourth, normalize all these eigenvectors from the last step, and plug everything in in the same places as with diagonalization from chapter 5.

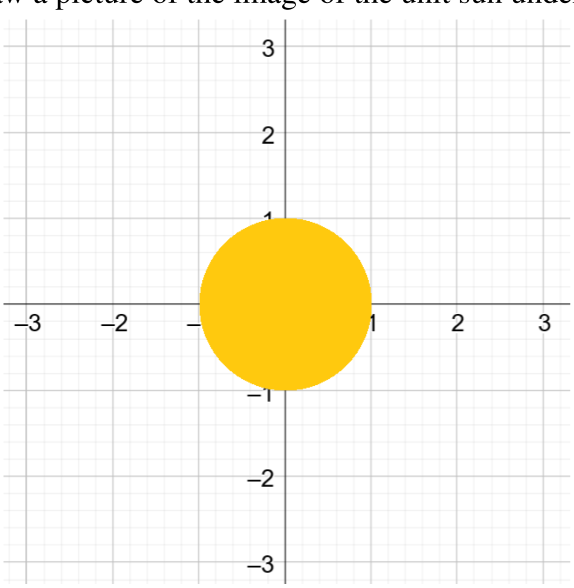
E. Using the spectral theorem to understand a linear transformation. We can interpret the factorization $A = PDP^T$ as saying that D is the matrix for the linear transformation $T(\mathbf{x}) = A\mathbf{x}$ in the orthonormal basis given by the columns of P.

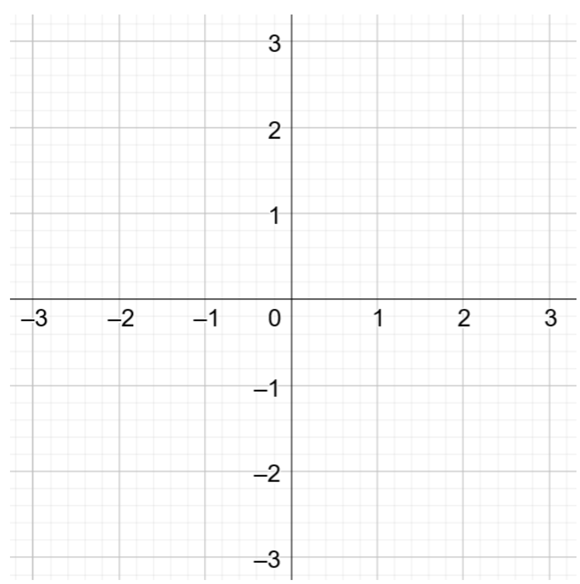
For the matrix $B = \begin{bmatrix} 1/2 & 3/2 \\ 3/2 & 1/2 \end{bmatrix}$, we have the orthogonal diagonalization $B = PDP^T$, with $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. We are going to use this to understand the linear transformation $T(\mathbf{x}) = B\mathbf{x}$.

(1) The two columns $\mathbf{u_1}, \mathbf{u_2}$ of P form an orthonormal basis for $\mathbb{R}^2$. Draw these vectors, and make axes in these two directions, marking off a tick mark each unit length.

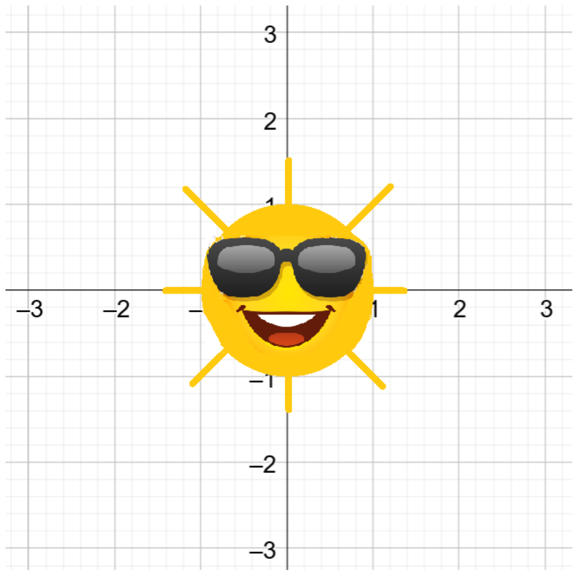
(2) Fill in the blanks: T corresponds to stretching by 2 in the ____ direction and __________________ over the ____ axis.

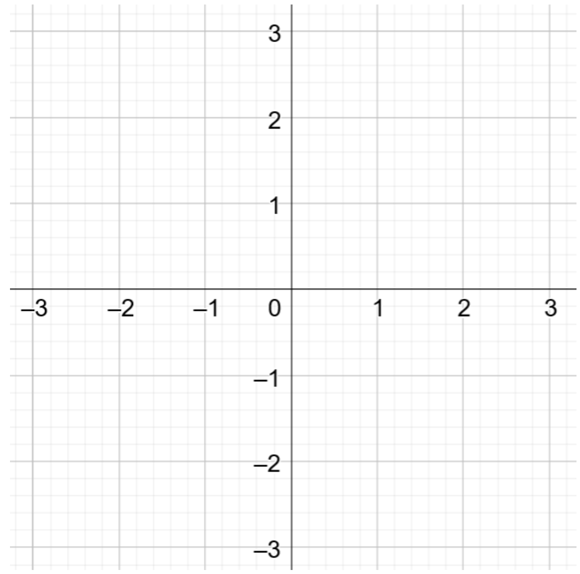
(3) Draw a picture of the image of the unit sun under the transformation T.





(4) Draw a picture of the image of the happy unit sun under the transformation T.





If A is an $m \times n$ matrix, the **singular values** of A are the *square roots* of the eigenvalues of A^TA. We denote these by σ_i, and we generally list them in decreasing order, and if an eigenvalue has multiplicity t, we repeat the corresponding singular value t times on the list.

A **singular value decomposition** (SVD) of an $m \times n$ matrix A is an expression $A = U\Sigma V^T$ where

$$\Sigma = \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_t & \\ \hline & 0 & & 0 \end{array}\right],$$

U is an $m \times m$ orthogonal matrix, and V is an $n \times n$ orthogonal matrix.

The values $\sigma_1, \dots, \sigma_t$ are the nonzero singular values of A, and the columns of V are an orthonormal basis of eigenvectors for A^TA.

F. Computing an SVD In this problem, we will compute a singular value decomposition for the matrix $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$.

(1) First, we compute an orthonormal diagonalization for A^TA. This involves all of the steps from problem D! An orthonormal diagonalization $A^TA = PDP^T$ is $P = \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ 2/3 & -1/3 & -2/3 \\ 2/3 & 2/3 & 1/3 \end{bmatrix}$ and $D = \begin{bmatrix} 360 & 0 & 0 \\ 0 & 90 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(2) Second, fill in the matrices Σ and V in the decomposition. The entries of Σ are the singular values, and the columns of V are the eigenvectors of A^TA in the corresponding order.

(3) Third, compute the columns of U as the normalized (to unit length) vectors $A\mathbf{v_i}$ for the columns $\mathbf{v_i}$ of V.

G. DECOMPOSITIONS AND INVERSES.

(1) If A is symmetric and invertible, and $A = PDP^T$ is an orthogonal diagonalization of A, then find an orthogonal diagonalization of A^{-1}.
(2) If A is invertible, and $A = U\Sigma V^T$ is a singular value decomposition of A, then find a singular value decomposition of A^{-1}.

H. KEY POINTS IN THESE DECOMPOSITIONS.

(1) If $\mathbf{v}$ is an eigenvector of A^TA, and $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, explain why $A\mathbf{u}$ and $A\mathbf{v}$ are orthogonal. This explains why the matrix U we make in the last step of the SVD is orthogonal.
(2) If A is symmetric, and $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors with different eigenvalues, why are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?[2]

I. SVD AND RANK.

(1) Explain why $\mathrm{Col}(A) = \mathrm{Null}(A^T)^\perp$.
(2) Explain why $\mathrm{Col}(A) \cap \mathrm{Null}(A^T) = \{\mathbf{0}\}$.
(3) Explain why $\mathrm{Null}(A) = \mathrm{Null}(A^TA)$.
(4) Explain why $\mathrm{rank}(A^TA) = \mathrm{rank}(A)$.
(5) Explain why the number of nonzero singular values of A is equal to the rank of A.

[2] Hint: $(A\mathbf{u}) \cdot \mathbf{v} = \mathbf{u}^T A^T \mathbf{v} = \mathbf{u}^T A \mathbf{v} = \mathbf{u} \cdot (A\mathbf{v})$.