Math 314. Week 10 worksheet (§5.3, §5.4, §5.5).

If A is an $n \times n$ matrix, then

$$\begin{array}{c} A = PDP^{-1} \\ \text{with } D \text{ diagonal} \end{array} \iff \begin{array}{ll} P = \begin{bmatrix} \mathbf{b_1} & \cdots & \mathbf{b_n} \end{bmatrix} & \text{where } \mathbf{b_1}, \dots, \mathbf{b_n} \text{ are } n \text{ linearly independent eigenvectors for } A \\ D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} & \text{where } \lambda_\mathbf{i} \text{ is the eigenvalue of } \mathbf{b_i}. \end{array}$$

DEFINITION: An $n \times n$ matrix A is **diagonalizable** if we can write $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

THEOREM: An $n \times n$ matrix A is diagonalizable if and only if the dimensions of its eigenspaces add up to n. When this happens, if you take a basis for each eigenspace, and combine these bases together, you get a set of n linearly independent eigenvectors for A.

THEOREM: If an $n \times n$ matrix has n distinct eigenvalues, then it is diagonalizable. However, not every diagonalizable $n \times n$ matrix has n distinct eigenvalues.

A. DIAGONALIZING A MATRIX. Let $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$. Our goal in this problem diagonalize A.

(1) Find the eigenvalues of A: compute the characteristic polynomial, and find its roots.
(2) For each eigenvalue, find a basis for the eigenspace. [1]
(3) Put together all of the bases for the different eigenspaces into one set; you should have 2 here. Stack them together to form P.
(4) Put the corresponding eigenvalues on the diagonal in the some order to get D.

B. DIAGONALIZING ANOTHER MATRIX? Let $B = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. Explain why the eigenvalues of B are 2 and 3. Find the dimensions of the $\lambda = 2$ and $\lambda = 3$ eigenspaces. Is B diagonalizable?

C. DIAGONALIZATION AND POWERS.

(1) If $D = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, find a formula for every[2] power D^n of D.
(2) If $D = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$, what happens to D^n as $n \to \infty$?
(3) If $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, what happens to D^n as $n \to \infty$?
(4) If $A = PDP^{-1}$, explain why $A^n = PD^nP^{-1}$ for every[3] n.
(5) If A is a 2×2 matrix, and all of its eigenvalues are larger than 1, what happens to A^n as $n \to \infty$?
(6) If A is a 2×2 matrix, and all of its eigenvalues are between 0 and 1, what happens to A^n as $n \to \infty$?

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[1] Remember that the λ-eigenspace is the null space of $A - \lambda I$.
[2] Hint: Start with $n = 2$, then $n = 3$,...
[3] Hint: Start with $n = 2$, then $n = 3$,...

DEFINITION: If $T : V \to V$ is a linear transformation, and $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is a basis for V, then the **$\mathcal{B}$-matrix of** T is the matrix such that $[T]_\mathcal{B} \cdot [\mathbf{v}]_\mathcal{B} = [T(\mathbf{v})]_\mathcal{B}$ for all $\mathbf{v} \in V$. It is given by the formula

$$[T]_\mathcal{B} = \begin{bmatrix}[T(\mathbf{b_1})]_\mathcal{B} & \cdots & T(\mathbf{b_n})]_\mathcal{B}\end{bmatrix}.$$

D. THE $\mathcal{B}$-MATRIX OF A LINEAR TRANSFORMATION ON POLYNOMIALS. Let P_3 be the vector space of all polynomials of degree at most 3. Consider the function $D : P \to P$ given by $D(f(t)) = \frac{df}{dt}$. We checked last time that this is a linear transformation. Take the basis $\mathcal{B} = \{t^3, t^2, t, 1\}$ for P_3. Find the $\mathcal{B}$-matrix for D. Is it diagonalizable?

E. THE $\mathcal{B}$-MATRIX OF A LINEAR TRANSFORMATION ON OTHER FUNCTIONS. Let $\mathrm{Fun}(\mathbb{R}, \mathbb{R})$ be the vector space of all real valued functions. Let $S = \mathrm{Span}\{\sin(x), \cos(x)\}$.

(1) Are $\sin(x)$ and $\cos(x)$ linearly independent?[4]
(2) Find a basis $\mathcal{B}$ for S. What is $\dim(S)$?
(3) Consider the linear transformation $D : S \to S$ given by $D(f(x)) = \frac{df}{dx}$. Find the $\mathcal{B}$-matrix for D.
(4) Find the $\mathcal{B}$-matrix for D^2.

F. THE $\mathcal{B}$-MATRIX OF A LINEAR TRANSFORMATION ON MATRICES. Consider the vector space $M_{2\times 2}$ of 2×2 matrices. Take the basis $\mathcal{B} = \left\{ \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} \right\}$. Let $F : M_{2\times 2} \to M_{2\times 2}$ be the function $F(X) = \begin{bmatrix}4 & -6\\ -2 & 3\end{bmatrix} X$.

(1) Show that the function F is a linear transformation.
(2) Find the $\mathcal{B}$-matrix for F.
(3) Find a basis for the null space of $[F]_\mathcal{B}$. Use this to find a basis for the kernel of F.
(4) Find a basis for the column space of $[F]_\mathcal{B}$. Use this to find a basis for the range of F.
(5) Let $G : M_{2\times 2} \to M_{2\times 2}$ be the function $G(X) = X + \begin{bmatrix}4 & -6\\ -2 & 3\end{bmatrix}$. Can you find the $\mathcal{B}$-matrix of G?

Recall from last time that $P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the matrix such that $P_{\mathcal{C}\leftarrow\mathcal{B}} \cdot [\mathbf{v}]_\mathcal{B} = [\mathbf{v}]_\mathcal{C}$ for all $\mathbf{v} \in V$. If $\mathcal{B}$ and $\mathcal{C}$ are two different bases for V, and $T : V \to V$ is a linear transformation, then $[T]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}} \cdot [T]_\mathcal{B} \cdot P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}$.

G. MATRICES FOR LINEAR TRANSFORMATIONS AND DIAGONALIZATION.

(1) Explain why the formula $[T]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}} \cdot [T]_\mathcal{B} \cdot P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}$ is true.
(2) If $V = \mathbb{R}^n$ and $\mathcal{C}$ is the standard basis $\{\mathbf{e_1}, \ldots, \mathbf{e_n}\}$ explain how $P_{\mathcal{C}\leftarrow\mathcal{B}}$ and $P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}$ are related to $P_\mathcal{B} = \begin{bmatrix}\mathbf{b_1} & \cdots & \mathbf{b_n}\end{bmatrix}$.
(3) If $V = \mathbb{R}^n$ and $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ are all eigenvectors for T with eigenvalues $\lambda_1, \ldots, \lambda_n$ (respectively) use the definition of $\mathcal{B}$-matrix to compute $[T]_\mathcal{B}$.
(4) In the setting of part (2) and (3), plug in all the pieces into the formula $[T]_\mathcal{C} = P_{\mathcal{C}\leftarrow\mathcal{B}} \cdot [T]_\mathcal{B} \cdot P^{-1}_{\mathcal{C}\leftarrow\mathcal{B}}$, and compare to the formula on the first page.
(5) Explain the following: two matrices are similar if they correspond to the same linear transformation in different bases.
(6) Explain the following: an $n \times n$ matrix A is diagonalizable if and only if there is a basis $\mathcal{B}$ of $\mathbb{R}^n$ such that the $\mathcal{B}$-matrix of the transformation $\mathbf{x} \mapsto A\mathbf{x}$ is a diagonal matrix.

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[4]Hint: if $a\sin(x) + b\cos(x) = 0$, plug in $x = 0$ and $x = \pi/2$.