Definitions for Math 314

- **basis** A set $\mathcal{B}$ of vectors in a vector space V is a BASIS for V if
  (i) $\mathcal{B}$ is linearly independent
  (ii) $\mathcal{B}$ spans V.
- **change-of-coordinates matrix**
  (1) The CHANGE-OF-COORDINATES MATRIX associated to a basis $\mathcal{B}$ of $\mathbf{R}^n$ is the matrix $P_{\mathcal{B}}$ such that $P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.
  (2) The CHANGE-OF-COORDINATES MATRIX associated to the pair of bases $\mathcal{B}$ and $\mathcal{C}$ of a vector space V is the matrix $P_{\mathcal{C}\leftarrow\mathcal{B}}$ such that $P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}]_{\mathcal{C}}$ for all $\mathbf{v} \in V$.
- **characteristic polynomial** The CHARACTERISTIC POLYNOMIAL of a matrix A is the polynomial function of λ given by $\det(A - \lambda I)$.
- **column space** The COLUMN SPACE of an $m \times n$ matrix A is the subspace of $\mathbb{R}^m$ spanned by the columns of A. We denote this by $\mathrm{Col}(A)$.
- **consistent** A linear system is CONSISTENT if it has a solution.
- **determinant** The DETERMINANT of an $n \times n$ matrix A is the number obtained from A by recursively applying Laplace expansion along rows or columns, and for 1×1 matrices is just the value of its entry. We write $\det(A)$ for the determinant of A.
- **diagonalizable** An $n \times n$ matrix is DIAGONALIZABLE if it is similar to a diagonal matrix.
- **dimension** The DIMENSION of a vector space V is the number of vectors in any basis for V. We denote this by $\dim(V)$.
- **eigenvector** If A is an $n \times n$ matrix, a nonzero vector $\mathbf{v} \in \mathbb{R}^n$ is an EIGENVECTOR for A if $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ.
- **eigenvalue**
  (1) If A is an $n \times n$ matrix, and $\mathbf{v}$ is an eigenvector for A, then λ is the EIGENVALUE associated to $\mathbf{v}$ if $A\mathbf{v} = \lambda\mathbf{v}$.
  (2) If A is an $n \times n$ matrix, then λ is an EIGENVALUE of A if $A\mathbf{v} = \lambda\mathbf{v}$ for some nonzero vector $\mathbf{v}$.
- **eigenspace** If A is an $n \times n$ matrix, and λ is an eigenvalue for A, the λ-EIGENSPACE of A is $\mathrm{Null}(A - \lambda I)$.
- **homogeneous** A linear system is HOMOGENEOUS if the constant part of each equation is zero.
- **inverse**
  (1) The INVERSE of an $n \times n$ matrix A, if it exists, is the matrix B such that $AB = BA = I_n$. We write A^{-1} for the inverse of A.
  (2) The INVERSE of a linear transformation $T : V \to W$, if it exists, is the linear transformation $U : W \to V$ such that $T \circ U = \mathrm{id}_W$ and $U \circ T = \mathrm{id}_V$. We write T^{-1} for the inverse of T.
- **invertible**
  (1) A matrix is INVERTIBLE if it has an inverse.
  (2) A linear transformation is INVERTIBLE if it has an inverse.
- **kernel** The KERNEL of a linear transformation $T : V \to W$ is the set of all $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{0}$.
- **least-squares solution** A vector $\mathbf{v}$ is a LEAST-SQUARES SOLUTION to a linear system $A\mathbf{x} = \mathbf{b}$ if $||\mathbf{b} - A\mathbf{v}|| \leq ||\mathbf{b} - A\mathbf{x}||$ for all vectors $\mathbf{x}$.

- **linear combination**
  (1) The LINEAR COMBINATION of the set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ with weights $c_1, \dots, c_t$ is the vector $c_1\mathbf{v_1} + \cdots + c_t\mathbf{v_t}$.
  (2) A LINEAR COMBINATION of the set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ is a vector of the form $c_1\mathbf{v_1}+\cdots+c_t\mathbf{v_t}$ for some $c_1, \dots, c_t$.
- **linear transformation** A function between two vector spaces $T : V \to W$ is a LINEAR TRANSFORMATION if the following conditions hold:
  (i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in V$ "T preserves addition"
  (ii) $T(c\mathbf{v}) = cT(\mathbf{v})$ for all $c \in \mathbb{R}, \mathbf{v} \in V$ "T preserves scalar multiplication."
- **linearly dependent** A set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ is LINEARLY DEPENDENT if $c_1\mathbf{v_1}+\cdots+c_t\mathbf{v_t} = \mathbf{0}$ has a solution other than $c_1 = \cdots = c_t = 0$.
- **linearly independent** A set of vectors $\{\mathbf{v_1}, \dots, \mathbf{v_t}\}$ is LINEARLY INDEPENDENT if the only solution to $c_1\mathbf{v_1} + \cdots + c_t\mathbf{v_t} = \mathbf{0}$ is the solution $c_1 = \cdots = c_t = 0$.
- **$\mathcal{B}$-matrix** If $\mathcal{B}$ is a basis for V, and $T : V \to V$ is a linear transformation, the $\mathcal{B}$-MATRIX of T is the matrix $[T]_\mathcal{B}$ such that $[T]_\mathcal{B}[\mathbf{v}]_\mathcal{B} = [T(\mathbf{v})]_\mathcal{B}$ for all $\mathbf{v} \in V$.
- **matrix transformation** A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a MATRIX TRANSFORMATION if there is an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.
- **null space** The NULL SPACE of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ consisting of all solutions $\mathbf{x}$ to the homogeneous linear system $A\mathbf{x} = \mathbf{0}$. We denote this by $\mathrm{Null}(A)$.
- **one-to-one** A function $f : V \to W$ is ONE-TO-ONE if $v \neq w$ implies $f(v) \neq f(w)$.
- **onto** A function $f : V \to W$ is ONTO if for every $w \in W$ can be written as $w = f(v)$ for some $v \in V$.
- **orthogonal**
  (1) Two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are ORTHOGONAL if $\mathbf{v} \cdot \mathbf{w} = 0$.
  (2) A set of vectors in $\mathbb{R}^n$ is ORTHOGONAL if each pair of vectors in the set is orthogonal.
  (3) A matrix is ORTHOGONAL if it is square, and its columns form an orthonomal set.
- **orthogonal complement** The ORTHOGONAL COMPLEMENT of a subspace $H \subseteq \mathbb{R}^n$ is the set of vectors in $\mathbb{R}^n$ that are orthogonal to every vector in H. It is denoted $H^\perp$.
- **orthonormal** A set of vectors is ORTHONORMAL if it is an orthogonal set of unit vectors.
- **projection** The PROJECTION of a vector $\mathbf{v}$ onto the subspace H of $\mathbb{R}^n$ is the unique vector $\hat{\mathbf{v}} \in H$ such that $\mathbf{v} - \hat{\mathbf{v}} \in H^\perp$. It is denoted $\mathrm{proj}_H(\mathbf{v})$.
- **range** The RANGE of a function $f : V \to W$ is the subset of W consisting of $f(v)$ for all $v \in V$.
- **rank** The RANK of a matrix A is the number of pivots of A in an echelon form for A.
- **similar** Two $n \times n$ matrices A and B are SIMILAR if there is an invertible matrix S such that $A = SBS^{-1}$.
- **solution** An ordered set of numbers $(a_1, \dots, a_n)$ is a SOLUTION to a linear system if each equation in the system holds true when a_i is substituted in for the ith variable.
- **solution set** The SOLUTION SET of a linear system is the set of all solutions to a linear system.
- **span** The SPAN of a set of vectors S is the set of all linear combinations of S. We write $\mathrm{Span}\{S\}$ for the span of S.
- **spans** A set of vectors SPANS a vector space if its span is that vector space.
- **standard matrix** The STANDARD MATRIX of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is the $m \times n$ matrix such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

- **subspace** A subset H of a vector space V is a SUBSPACE if the following three conditions hold:
  (i) $\mathbf{0} \in H$
  (ii) $\mathbf{u}, \mathbf{v} \in H \implies \mathbf{u} + \mathbf{v} \in H$ "H is closed under addition"
  (iii) $c \in \mathbb{R},\ \mathbf{v} \in H \implies c\mathbf{v} \in H$ "H is closed under scalar multiplication."
- **symmetric** A matrix A is symmetric if $A^T = A$.
- **vector**
  (1) A VECTOR in $\mathbb{R}^n$ is an n-tuple of real numbers, which we write as $\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.
  (2) A VECTOR in a vector space V is an element of the set V.
- **vector space** A VECTOR SPACE is a set V with two operations, which we call addition $V \times V \to V$ and scalar multiplication $\mathbb{R} \times V \to V$, that are subject to a number of conditions.
- **vector of $\mathcal{B}$-coordinates** If $\mathcal{B} = \{\mathbf{b_1}, \ldots, \mathbf{b_n}\}$ is a basis for V, the VECTOR OF $\mathcal{B}$-COORDINATES of a vector $\mathbf{v} \in V$ is the vector $[\mathbf{v}]_{\mathcal{B}} \in \mathbb{R}^n$ determined by the rule

$$[\mathbf{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \iff \mathbf{v} = c_1\mathbf{b_1} + \cdots + c_n\mathbf{b_n}$$

Notation

- $[T]_{\mathcal{B}}$: $\mathcal{B}$-matrix
- $[\mathbf{x}]_{\mathcal{B}}$: vector of $\mathcal{B}$-coordinates
- $\text{Span}(S)$: span
- $\dim(V)$: dimension
- $\text{proj}_W(\mathbf{x})$: projection
- $P_{\mathcal{B}}$: change-of-coordinates matrix
- $P_{\mathcal{C} \leftarrow \mathcal{B}}$: change-of-coordinates matrix
- $H^{\perp}$: orthogonal complement
- $\text{Col}(A)$: column space
- $\text{Null}(A)$: null space
- $\text{Ker}(T)$: kernel
- A^{-1}: inverse