# Sheet 13 ~~solutions~~

A. 1) $\times$ ✓

$\times$ $\times$

2) yes: $(A^TA)^T = A^T A^{TT} = A^TA$

3) $$A^TA = \begin{bmatrix} \underline{a}_1^T \\ \vdots \\ \underline{a}_n^T \end{bmatrix} \begin{bmatrix} \underline{a}_1 & \cdots & \underline{a}_n \end{bmatrix} = \begin{bmatrix} \underline{a}_1 \cdot \underline{a}_1 & \underline{a}_1 \cdot \underline{a}_2 & \cdots & \underline{a}_1 \cdot \underline{a}_n \\ \underline{a}_2 \cdot \underline{a}_1 & \underline{a}_2 \cdot \underline{a}_2 & \cdots & \\ \vdots & \vdots & \ddots & \\ \underline{a}_n \cdot \underline{a}_1 & & & \underline{a}_n \cdot \underline{a}_n \end{bmatrix}$$

4) no: $[A^TA]_{11} = \underline{a}_1 \cdot \underline{a}_1 \geq 0$ always,

~~so~~ $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ is not of the form $A^TA$.

B. 1) $Q$ orthogonal $\iff$ $\underline{q}_i \cdot \underline{q}_j = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

$$\iff \begin{bmatrix} \underline{q}_1 \cdot \underline{q}_1 & \underline{q}_1 \cdot \underline{q}_2 & \cdots \\ \underline{q}_2 \cdot \underline{q}_1 & \underline{q}_2 \cdot \underline{q}_2 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 1 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$$

$$\iff Q^TQ = I_n$$

2) If $Q^TQ = I$, since $Q, Q^T$ are square, $Q^T = Q^{-1}$.

3) - rotation yes
- stretching no
- reflection yes

C. 1) F
2) T
3) F (real eigenvalue)
4) T
5) F
6) F
7) T

D. 1) Find the roots of $\det(B - \lambda I)$
2) Find bases for the nullspaces of $B - 27I$, $B + 9I$.

3) $\begin{bmatrix}2\\1\\0\end{bmatrix}$, $\begin{bmatrix}2\\0\\1\end{bmatrix} - \dfrac{\begin{bmatrix}2\\0\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\0\end{bmatrix}}{\begin{bmatrix}2\\1\\0\end{bmatrix}\cdot\begin{bmatrix}2\\1\\0\end{bmatrix}}\begin{bmatrix}2\\1\\0\end{bmatrix} = \begin{bmatrix}2/5\\-4/5\\1\end{bmatrix}$

✓

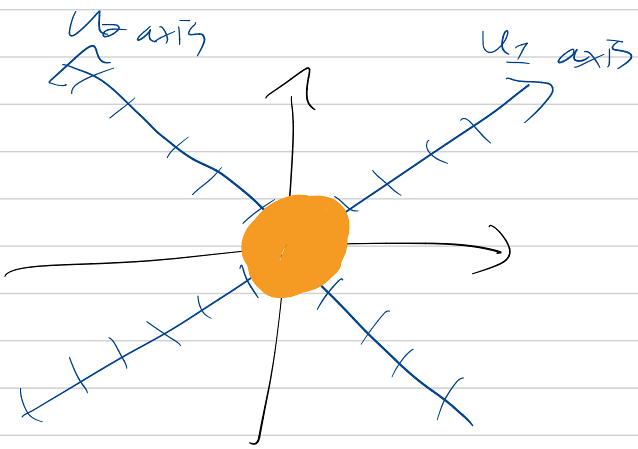
Any vector in the $\lambda = -9$ eigenspace is orthogonal with any vector in the $\lambda = 27$ eigenspace, by a theorem.

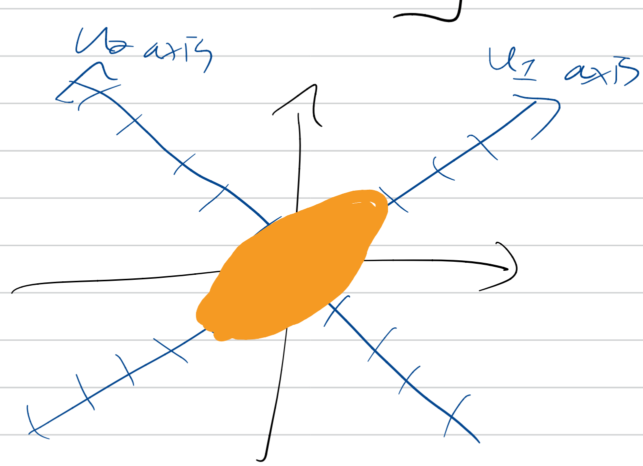
Also, can just check $\begin{bmatrix}-1/2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}2\\1\\0\end{bmatrix} = 0$

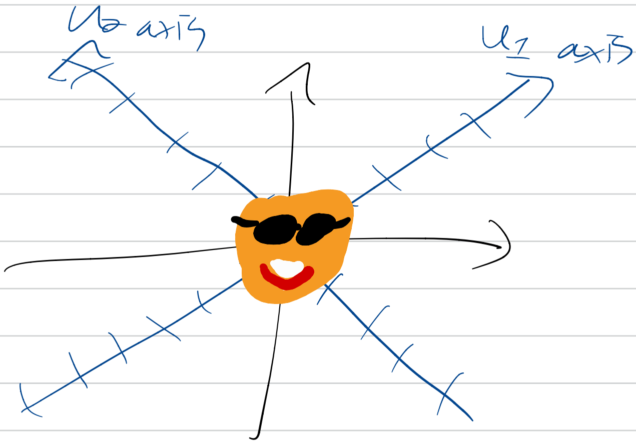
$\begin{bmatrix}-1/2\\1\\1\end{bmatrix}\cdot\begin{bmatrix}2/5\\-4/5\\1\end{bmatrix} = 0.$

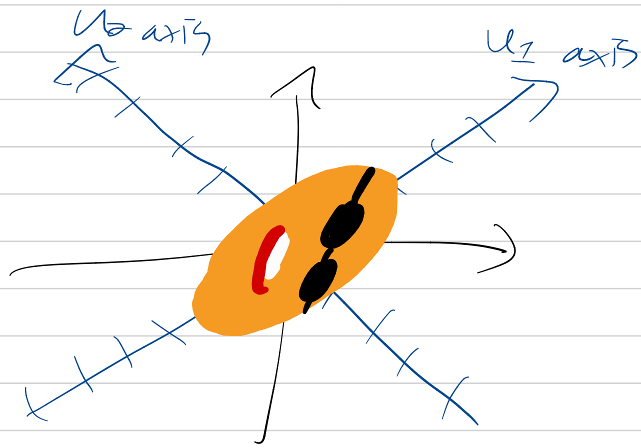
4) $P = \begin{bmatrix}2/\sqrt{5} & 2/3\sqrt{5} & -1/3\\ 1/\sqrt{5} & -4/3\sqrt{5} & 2/3\\ 0 & \sqrt{5}/3 & 2/3\end{bmatrix}$ $D = \begin{bmatrix}27 & 0 & 0\\ 0 & 27 & 0\\ 0 & 0 & -9\end{bmatrix}$

E.








stretch by 2 in $\underline{u_1}$ direction
& flip over the $\underline{u_1}$-axis.

F. 2) $\Sigma = \begin{bmatrix} \sqrt{360} & 0 & 0 \\ 0 & \sqrt{90} & 0 \end{bmatrix}$ $V = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$

3) $A\frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$ $A\frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -9 \end{bmatrix}$

$\begin{bmatrix} 3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}$ $\begin{bmatrix} 1/\sqrt{10} \\ -3/\sqrt{10} \end{bmatrix}$

$U = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & 1 \\ 1 & -3 \end{bmatrix}$.

G. 1) $(PDP^T)^{-1} = (P^T)^{-1} D^{-1} P^{-1}$

$= P D^{-1} P^T$

so just replace eigvals in $D$ by reciprocals.

2) $(U\Sigma V^T)^{-1} = (V^T)^{-1} \Sigma^{-1} U^{-1}$

$= V \Sigma^{-1} U^T$

so switch $U$ & $V$ ~~and~~ and replace singular values by reciprocals

H. 1) $A\underline{u} \cdot A\underline{v} = \underline{u}^T A^T A \underline{v} = \underline{u}^T \lambda \underline{v} = \lambda \underline{u}^T \underline{v}$

$= \lambda\, \underline{u} \cdot \underline{v} = 0.$

2) $A\underline{u} \cdot \underline{v} = \underline{u}^T A^T \underline{v} = \underline{u}^T A \underline{v} = \underline{u}^T \lambda_2 \underline{v}$

$= \lambda_2 (\underline{u} \cdot \underline{v})$

$(\lambda_1 \underline{u}) \cdot \underline{v} = \lambda_1 (\underline{u} \cdot \underline{v})$

Since $\lambda_1 \neq \lambda_2$, $\underline{u} \cdot \underline{v} = 0$.

I. 1) $Col(A) = Col((A^T)^T) = Null(A^T)^\perp$.

2) If $\underline{u} \in Col(A) \cap Null(A^T)$

$= Null(A^T)^\perp \cap Null(A^T)$,

then $\underline{u} \cdot \underline{u} = 0$, so $\underline{u} = \underline{0}$.

3) If $A^T A \underline{x} = 0$, then

$A\underline{x} \in Null(A^T) \cap Col(A)$, so

$A\underline{x} = 0$, so $\underline{x} \in Null(A)$.

If $A\underline{x} = \underline{0}$, then $A^T A \underline{x} = A^T \underline{0} = \underline{0}$.

4) They have same # columns &

same rank, so good by rank nullity.

5) # nonzero sing values = rank($A^T A$) = rank($A$).