A.1) characteristic polynomial

$$= \det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 2.$$

eigenvalues are $\lambda = 2, \lambda = -1$

2) $\lambda = 2 \rightsquigarrow \text{Null}\left(\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$

basis $\left\{\begin{bmatrix} 1/2 \\ 1 \end{bmatrix}\right\}$

$\lambda = -1 \rightsquigarrow \text{Null}\left(\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$

basis $\left\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right\}$.

3) $P = \begin{bmatrix} 1/2 & -1 \\ 1 & 1 \end{bmatrix}$

4) $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

B. triangular matrix

→ eigenvalues on diagonal

$\lambda=2$ eigenspace $= \text{Null}\left(\begin{bmatrix}0&1&0\\0&0&1\\0&0&1\end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}\right\}$

1-dimensional

$\lambda=3$ eigenspace $= \text{Null}\left(\begin{bmatrix}-1&1&0\\0&-1&1\\0&0&0\end{bmatrix}\right) = \text{Span}\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}\right\}$

1-dimensional

C. 1) $D^n = \begin{bmatrix}a^n & 0\\0 & b^n\end{bmatrix}$

2) $D^n \to 0$

3) D^n gets larger & larger

4) $(PDP^{-1})^n = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})$

$= PD^nP^{-1}$.

5) gets ~~biggerd~~ bigger

6) gets smaller ~~&~~ smaller

D. $[D]_\beta = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ not diagonalizable

E. 1) yes : $a\sin(x) + b\cos(x) = 0$ (0 function)

$\Rightarrow a = b = 0$.

2) $\beta = \{\sin(x), \cos(x)\}$. $\dim(\beta) = 2$.

3) $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ 4) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$.

F. 1) $F(X+Y) = A(X+Y) = AX + AY$

$= F(X) + F(Y)$

$F(cX) = A(cX) = cAX = cF(X)$.

2) $\begin{bmatrix} 4 & 0 & -6 & 0 \\ 0 & 4 & 0 & -6 \\ -2 & 0 & 3 & 0 \\ 0 & -2 & 0 & 3 \end{bmatrix}$

3) ~~A~~ basis for $\mathrm{Null}([F]_B)$

is $\left\{ \begin{bmatrix} 3/2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$.

A basis for $\ker(F)$

is $\left\{ \begin{bmatrix} 3/2 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3/2 \\ 0 & 1 \end{bmatrix} \right\}$.

4) A basis for $Col([F]_\beta)$

is $\left\{ \begin{bmatrix} 4 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix} \right\}$.

A basis for Range(F) is

$\left\{ \begin{bmatrix} 4 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 0 & -2 \end{bmatrix} \right\}$.

5) No, it's ~~not~~ a linear transformation

6. 1) $\left(P_{e \leftarrow \beta} \cdot [T]_\beta \cdot P_{e \leftarrow \beta}^{-1}\right) [x]_e$

$$= P_{e \leftarrow \beta} [T]_\beta \cdot [x]_\beta$$

$$= P_{\mathcal{C} \leftarrow \mathcal{B}} \left([T(\underline{x})]_{\mathcal{B}} \right)$$

$$= [T(\underline{x})]_{\mathcal{C}}.$$

Also $[T]_{\mathcal{C}} \cdot [\underline{x}]_{\mathcal{C}} = [T(\underline{x})]_{\mathcal{C}}$.

This is true for every $\underline{x}$, so they must be equal.

2) $P_{\mathcal{C} \leftarrow \mathcal{B}} = P_{\mathcal{B}}$

and $P_{\mathcal{C} \leftarrow \mathcal{B}}^{-1} = P_{\mathcal{B}}^{-1}$.

3) $T(\underline{b_i}) = \lambda_i \underline{b_i}$ for every i, so

$$[T]_{\mathcal{B}} \cdot \underline{e_i} = [T]_{\mathcal{B}} \cdot [\underline{b_i}]_{\mathcal{B}} = [T(\underline{b_i})]_{\mathcal{B}}$$

$$= [\lambda_i \underline{b_i}]_{\mathcal{B}} = \lambda_i \underline{e_i}.$$

This means $[T]_\beta = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$.

4) $\underset{\substack{\shortparallel \\ \text{standard} \\ \text{matrix of } T}}{[T]_\mathcal{E}} = \underbrace{P_{\mathcal{E}\leftarrow\beta} \cdot [T]_\beta \cdot P_{\mathcal{E}\leftarrow\beta}^{-1}}_{\begin{bmatrix} \underline{b_1} & \cdots & \underline{b_n} \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} \underline{b_1} & \cdots & \underline{b_n} \end{bmatrix}^{-1}}$.

Same as formula.

5) Follows from

$$[T]_\mathcal{E} = P_{\mathcal{E}\leftarrow\beta} \cdot [T]_\beta \cdot P_{\mathcal{E}\leftarrow\beta}^{-1}.$$

6) Using (5), similar to diagonal matrix means diagonal in some basis.