Learning Objectives:

- Understand how to construct a singular value decomposition of a matrix

The Singular Values of an $m \times n$ Matrix

Definition: If A is an $m \times n$ matrix, the **singular values** of A are the ________________________

of the eigenvalues of __________, denoted by $\sigma_1, \ldots, \sigma_n$, and they are arranged in decreasing order.

That is, $\sigma_i =$ _______ for $1 \leq i \leq n$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$.

Note: The singular values of A are the ______________ of the vectors ________________________.

Theorem 7.9. Suppose $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A^TA, arranged so that the corresponding eigenvalues of A^TA satisfy ___________________________

and suppose A has _____ nonzero singular values.

Then $\{A\mathbf{v}_1, \ldots, A\mathbf{v}_r\}$ is an __________________________________ for Col A, and rank $A = r$.

Remark: In practice, the most reliable way to estimate the rank of a large matrix A is to count the number of nonzero singular values.

The Singular Value Decomposition

The decomposition of A involves an $m \times n$ "diagonal" matrix Σ of the form

$$\Sigma =$$

where D is an __________ diagonal matrix for some r not exceeding the smaller of m and n.

Theorem 7.10 (The Singular Value Decomposition).

Let A be an $m \times n$ matrix with rank r. Then there exists an $m \times n$ matrix Σ (as shown above) for which the diagonal entries in D are the first ______________________________ of A, $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$, and there exists an $m \times m$ __________________ matrix U and an $n \times n$ __________________ matrix V such that

$$A =$$

How to find a singular value decomposition

If A is an $m \times n$ matrix...

Step 1: Find an orthogonal diagonalization of $A^T A$. This means to find all the eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (each listed as many times as its multiplicity) and an orthonormal basis $\mathbf{v_1}, \ldots, \mathbf{v_n}$ of $\mathbb{R}^n$ consisting of eigenvectors of $A^T A$ (that match up with the eigenvalues, so $\mathbf{v_i}$ has eigenvalue λ_i).

Step 2: Fill in Σ and V. We compute the singular values $\sigma_i = \sqrt{\lambda_i}$, and fill them into Σ, and fill in the columns of V with the corresponding orthonormal basis of eigenvectors for $A^T A$.

Step 3: Construct U. The first r $(= \text{rank}(A))$ columns of U are the normalized (unit length) vectors obtained from $A\mathbf{v_1}, \ldots, A\mathbf{v_r}$. If $r < m$, then use Gram-Schmidt to complete an orthonormal basis for $\mathbb{R}^m$.