Learning Objectives:

- Determine whether a matrix is symmetric
- Understand how to orthogonally diagonalize a symmetric matrix

Diagonalization of Symmetric Matrices

Our goal in this section is to connect orthogonality with our knowledge of eigenvalues.

Definition: A matrix A is **symmetric** if ______________.

Note that symmetric matrices are necessarily ______________________.

Example Determine if the following matrices are symmetric.

(a) $\begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 5 & 0 \\ 2 & 0 & 4 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 2 & -1 \\ 3 & 2 & -1 & 1 \\ 2 & -1 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 6 & 2 \\ -2 & 0 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & -1 & 2 \\ 0 & 3 & -1 \\ 2 & 0 & 3 \end{bmatrix}$

(f) $\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$

Theorem 7.1.
If A is symmetric, then any two eigenvectors from different eigenspaces are _____________________.

Definition: An $n \times n$ matrix A is said to be **orthogonally diagonalizable** if there exists an

_________________________ matrix P and a _____________________ matrix D such that

$$A =$$

Note: In general, it can be difficult to determine whether a matrix is diagonalizable.

Theorem 7.2. An $n \times n$ matrix A is orthogonally diagonalizable if and only if ________________

____.

The Spectral Theorem

Theorem 7.3 (The Spectral Theorem for Symmetric Matrices).

An $n \times n$ symmetric matrix A has the following properties:

(a) A has _______ real eigenvalues, counting multiplicities.

(b) The dimension of the eigenspace for each eigenvalue λ equals the _____________________ of λ as a root of the characteristic equation.

(c) The eigenspaces are mutually orthogonal, in the sense that _____________________________ corresponding to different eigenvalues are orthogonal.

(d) A is orthogonally diagonalizable.

Note: The set of eigenvalues of a matrix A is sometimes called the **spectrum** of A.

Spectral Decomposition

Suppose $A = PDP^{-1}$, where the columns of P are orthonormal eigenvectors $\mathbf{u}_1, \dots, \mathbf{u}_n$ of A and the corresponding eigenvalues $\lambda_1, \dots, \lambda_n$ are the diagonal entries of the diagonal matrix D. Then, since $P^{-1} = P^T$,

$$A = \lambda_1 \mathbf{u}_1 \mathbf{u}_1^T + \cdots + \lambda_n \mathbf{u}_n \mathbf{u}_n^T.$$