Learning Objectives:

- Understand how to find a least-squares solution of $A\mathbf{x} = \mathbf{b}$
- Understand how to find the least-squares error of a least-squares solution

Least-Squares Problems

Question: What do we do when $A\mathbf{x} = \mathbf{b}$ has no solution $\mathbf{x}$?

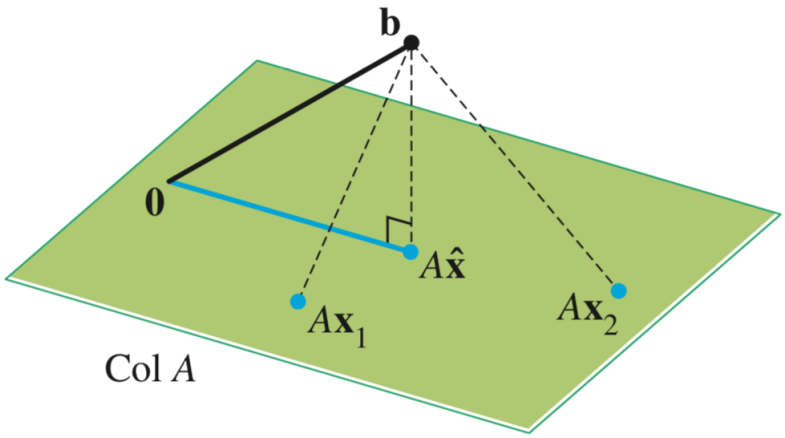
Answer: Find $\hat{\mathbf{x}}$ such that $A\hat{\mathbf{x}}$ is as close as possible to $\mathbf{b}$.

That is, we want to minimize $||\mathbf{b} - A\hat{\mathbf{x}}||$ (which is why these are called least-squares problems).

Definition: If A is an $m \times n$ matrix and $\mathbf{b}$ is in $\mathbb{R}^m$, a _____ solution of $A\mathbf{x} = \mathbf{b}$ is an $\hat{\mathbf{x}}$ in $\mathbb{R}^n$ such that

$$||\mathbf{b} - A\hat{\mathbf{x}}|| \leq \qquad \text{for all } \mathbf{x} \text{ in } \mathbb{R}^n.$$

The least squares error of the least squares solution is the value $||\mathbf{b} - A\hat{\mathbf{x}}||$.



Theorem 6.13. The set of least-squares solutions of $A\mathbf{x} = \mathbf{b}$ is the (nonempty) set of all solutions of the normal equations _____.

Example. Find a least-squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}$$

Theorem 6.14. Let A be an $m \times n$ matrix. The following statements are equivalent:

(a) The equation $A\mathbf{x} = \mathbf{b}$ has a _____ least-squares solution for each $\mathbf{b}$ in $\mathbb{R}^m$.

(b) The columns of A are _____.

(c) The matrix A^TA is _____.

When these statements are true, the least-squares solution $\hat{\mathbf{x}}$ is given by

$$\hat{\mathbf{x}} =$$

Example. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) Determine if A^TA is invertible.

(b) Find a least-squares solution of the inconsistent system $A\mathbf{x} = \mathbf{b}$.

(c) Determine the least-squares error in the least-squares solution of $A\mathbf{x} = \mathbf{b}$.