Learning Objectives

- Use the Gram-Schmidt process to construct an orthogonal basis for any nonzero subspace of $\mathbb{R}^n$
- Understand how to find a QR factorization of a matrix with linearly independent columns

The Gram-Schmidt Process

Our goal in this section is to develop a process for producing orthogonal bases for subspaces in $\mathbb{R}^n$.

Example: Suppose $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for a subspace W of $\mathbb{R}^4$. Construct an orthogonal basis for W.

Solution: We may as well take our first vector to be the same:

$$\mathbf{v}_1 = \mathbf{x}_1$$

We should use $\mathbf{x}_2$ and $\mathbf{v}_1$ to construct our second vector, $\mathbf{v}_2$, and we can use the Orthogonal Decomposition Theorem (Theorem 6.8) to do this so that $\mathbf{v}_2$ and $\mathbf{v}_1$ are orthogonal:

$$\mathbf{v}_2 = \mathbf{x}_2 - \operatorname{proj}_{\mathbf{v}_1} \mathbf{v}_1 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$$

For our third vector, we can use the same idea: we can project $\mathbf{x}_3$ onto the plane $W_2 = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and subtract that result from $\mathbf{x}_3$ using Theorem 6.8:

$$\mathbf{v}_3 = \mathbf{x}_3 - \operatorname{proj}_{W_2} \mathbf{x}_3 = \mathbf{x}_3 - \hat{\mathbf{x_3}} = \mathbf{x}_3 - \left(\frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2\right) = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2$$

Theorem 6.11: The Gram-Schmidt Process
Given a basis $\{\mathbf{x}_1, \dots, \mathbf{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$\mathbf{v}_1 =$

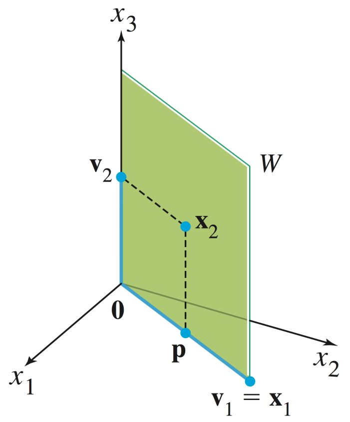
$\mathbf{v}_2 =$

$\mathbf{v}_3 =$

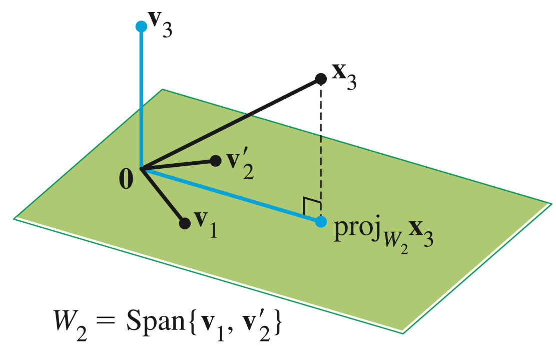
$\vdots$

$\mathbf{v}_p =$

Then $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is an orthogonal basis for W, and



How to construct $\mathbf{v}_2$ from $\mathbf{x}_2$ and $W_1 = \text{Span}\{\mathbf{v}_1\} = \text{Span}\{\mathbf{x}_1\}$



How to construct $\mathbf{v}_3$ from $\mathbf{x}_3$ and $W_2 = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$

Example: Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$, where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$.

Construct an orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for W.

Solution: $\mathbf{v}_1 = \mathbf{x}_1$, and $\mathbf{v}_2 = \mathbf{x}_2 - \text{proj}_{\mathbf{v}_1} \mathbf{x}_2 = \mathbf{x}_2 - \dfrac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 =$

(since $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 \cdot \mathbf{v}_1 =$ and $\mathbf{v}_1 \cdot \mathbf{v}_1 =$)

So, an orthogonal basis for W is:

Orthonormal Bases

An orthonormal basis can be constructed from an orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ by ______________ (i.e., scaling) ____________.

Example: Let $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -5 \\ 1 \\ 5 \\ 7 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 8 \end{bmatrix}$, and let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$.

(a) How could we show that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ is a basis for W, which is a subspace of $\mathbb{R}^4$?

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(b) Find an orthogonal basis for W.

$\mathbf{v}_1 = \mathbf{x}_1$

$$\mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 =$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 =$$

$$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \left\{ \begin{bmatrix} 3 \\ 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right\} \text{ is an orthogonal basis for } W.$$

(c) Find an orthonormal basis for W.

We need to normalize $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to make an orthonormal basis $\{\mathbf{v}'_1, \mathbf{v}'_2, \mathbf{v}'_3\}$:

$$\mathbf{v}'_1 = \frac{1}{\|\mathbf{v}_1\|}\mathbf{v}_1 = \begin{bmatrix} 3/\sqrt{20} \\ 1/\sqrt{20} \\ -1/\sqrt{20} \\ 3/\sqrt{20} \end{bmatrix} \qquad \mathbf{v}'_2 = \frac{1}{\|\mathbf{v}_2\|}\mathbf{v}_2 = \qquad \mathbf{v}'_3 =$$

QR Factorization of Matrices

If an $m \times n$ matrix A has linearly independent columns, then applying the Gram-Schmidt process (with normalizations) to the ______________ of A produces an extremely useful factorization for A.

Theorem 6.12: The QR Factorization

If A is an $m \times n$ matrix with ______________________________ columns, then A can be factored as

where:

- _____ is an $m \times n$ matrix whose columns form an __________________ basis for Col A, and
- _____ is an $n \times n$ ____________________________________ matrix with positive entries on its diagonal.

Note: If the columns of Q form an orthonormal basis for $\operatorname{Col} A$, then

$$Q^T Q =$$

Therefore, we can find R by

$$R = IR = Q^T(QR) = Q^T A$$