Learning Objectives

- Understand how to find the orthogonal projection of a vector in $\mathbb{R}^n$ onto a subspace
- Understand how to find the closest point in a subspace W to a vector in $\mathbb{R}^n$
- Understand how to find the distance from a subspace to a vector in $\mathbb{R}^n$

Orthogonal Projections

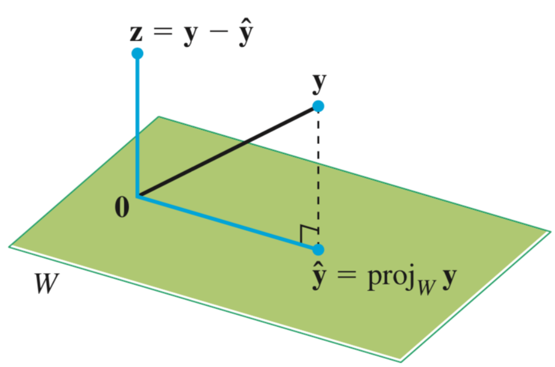
Theorem 6.8: The Orthogonal Decomposition Theorem
Let W be a subspace of $\mathbb{R}^n$. Then each $\mathbf{y}$ in $\mathbb{R}^n$ can be uniquely represented in the form

$\mathbf{y} =$ where ______ is in W and ______ is in $W^{\perp}$.

In fact, if $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal basis of W, then

Remark: This decomposition is easy if we have an ____.
We will discover in Section 6.4 that every nonzero subspace of $\mathbb{R}^n$ has an ________________________ ______________, and learn how to construct one.

Example: Let $\mathbf{u}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{y} = \begin{bmatrix} 0 \\ 3 \\ 10 \end{bmatrix}$. Observe that $\{\mathbf{u}_1, \mathbf{u}_2\}$ is an orthogonal basis for $W = \text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$. Write $\mathbf{y}$ as the sum of a vector in W and a vector orthogonal to W.



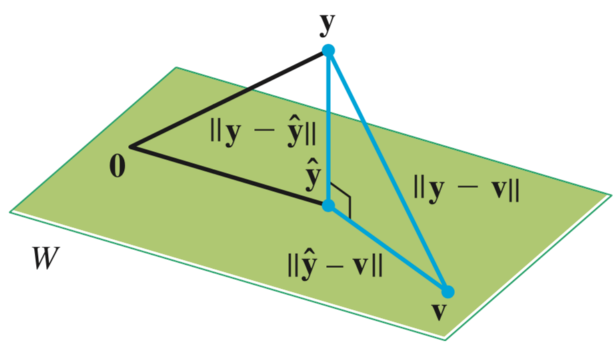
Definition: The vector $\hat{\mathbf{y}}$ in Theorem 6.8 is called the **orthogonal** ____________________ of ______ onto ______, and often is written as _____________. (See picture above.)

Properties of Orthogonal Projections

Theorem 6.9: The Best Approximation Theorem
Let W be a subspace of $\mathbb{R}^n$, $\mathbf{y}$ be any vector in $\mathbb{R}^n$, and $\hat{\mathbf{y}}$ be the orthogonal projection of $\mathbf{y}$ onto W. Then $\hat{\mathbf{y}}$ is the __________________________ in W to $\mathbf{y}$, in the sense that

for all $\mathbf{v} \neq \hat{\mathbf{y}}$ in W.

Proof: Use the Pythagorean Theorem.



Remarks:

- The vector $\hat{\mathbf{y}}$ in Theorem 6.9 is called the **best approximation to** ______ by elements of ______.
- If $\mathbf{y}$ is in W, then $\hat{\mathbf{y}} =$ ______.
- If we need to replace a vector $\mathbf{y}$ with a vector $\mathbf{v}$ in W, then error is minimized when $\mathbf{v} =$ ______.

Theorem 6.10: If $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthonormal basis for a subspace W of $\mathbb{R}^n$, then

$$\text{proj}_W\, \mathbf{y} =$$

If $U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_p \end{bmatrix}$, then

$$\text{proj}_W\, \mathbf{y} = \qquad\qquad\qquad \text{for all } \mathbf{y} \text{ in } \mathbb{R}^n.$$