Learning Objectives

- Determine whether a set is orthogonal
- Understand how to decompose a vector in $\mathbb{R}^n$ into orthogonal components
- Determine whether a set is orthonormal

Orthogonal Sets

Definition: A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is an **orthogonal set** if each pair of ________________________________ from the set is ______________________, that is, if ______________________ whenever $i \neq j$.

Example: Determine if $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthogonal set, where

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Theorem 6.4: If $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$ and $W = \text{Span}\,\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$, then S is _________________________________ and hence is a ________________ for W.

Definition: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an ________________________________.

Why Having an Orthogonal Basis is Nice

Example: Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and $\mathbf{y}$ be a vector in W. Find weights $c_1, \ldots, c_p$ such that

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p.$$

Theorem 6.5: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\mathbf{y}$ in W, the weights in the linear combination

$$\mathbf{y} =$$

are given by

$$c_j =$$

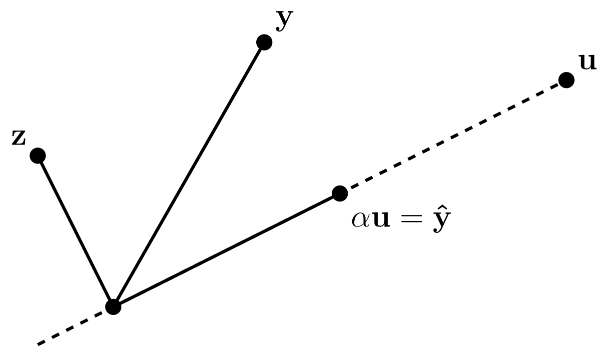
Example: From the example on page 1 and Theorem 6.4, we have that $S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis for $\mathbb{R}^3$. Express the vector $\mathbf{y} = \begin{bmatrix} 3 \\ 7 \\ 4 \end{bmatrix}$ as a linear combination of the vectors in S.

Orthogonal Projections

Let $\mathbf{u}$ be a nonzero vector in $\mathbb{R}^n$, and suppose we want to write a vector $\mathbf{y}$ in $\mathbb{R}^n$ as

$$\mathbf{y} = (\text{a scalar multiple of } \mathbf{u}) + (\text{a scalar multiple of a vector orthogonal to } \mathbf{u})$$

Thus, we want to write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$, where $\hat{\mathbf{y}} = \alpha\mathbf{u}$ for some scalar α and $\mathbf{z}$ is orthogonal to $\mathbf{u}$.



Our goal is to find α.

Note: The vector $\hat{\mathbf{y}}$ is called the ____________________________ **of y onto u**.

The vector $\mathbf{z}$ is called the **component of y** ____________________ **to u**.

Orthonormal Sets

Definition: A set of vectors $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is called an **orthonormal set** if it is an orthogonal set of ____________________. If $W = \text{Span}\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$, then $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ is an **orthonormal** ____________ for W.

Example: Let $U = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3\end{bmatrix}$ where $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is an orthonormal set in $\mathbb{R}^m$.

Compute U^TU.

Theorem 6.6: An $m \times n$ matrix U has orthonormal columns if and only if ______________.

Theorem 6.7: Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be vectors in $\mathbb{R}^n$. Then

(a) $||U\mathbf{x}|| =$

(b) $(U\mathbf{x}) \cdot (U\mathbf{y}) =$

(c) $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if

Definition: An **orthogonal matrix** is a square invertible matrix U such that ____________. Such a matrix has ________________________ columns.

Warning: A better name for this would have been *orthonormal matrix*, but we are stuck with the terminology.