Learning Objectives

- Understand how to find the inner product of two vectors in $\mathbb{R}^n$
- Understand how to find the length of a vector in $\mathbb{R}^n$
- Understand how to normalize a vector in $\mathbb{R}^n$
- Understand how to find the distance between two vectors in $\mathbb{R}^n$
- Determine whether two vectors in $\mathbb{R}^n$ are orthogonal to each other

The Inner Product

Definition: Let $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$.

The **inner product** of $\mathbf{u}$ and $\mathbf{v}$ is defined as

$$\mathbf{u} \cdot \mathbf{v} =$$

Example: Compute $\mathbf{u} \cdot \mathbf{v}$ and $\mathbf{v} \cdot \mathbf{u}$ for $\mathbf{u} = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix}$.

Theorem 6.1: Let $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ be vectors in $\mathbb{R}^n$, and let c be a scalar. Then

(a) $\mathbf{u} \cdot \mathbf{v} =$ __________

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} =$ ____________________

(c) $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v}) =$ ______________

(d) $\mathbf{u} \cdot \mathbf{u} \geq$ _______, and $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} =$ _______

Remark: Properties (b) and (c) can be combined to get the following rule:

The Length of a Vector

Let $\mathbf{v}$ be the vector in $\mathbb{R}^n$ whose entries are $v_1, v_2, \dots, v_n$.

Definition: The **length** (or **norm**) of $\mathbf{v}$ is the nonnegative scalar $||\mathbf{v}||$ defined by

$$||\mathbf{v}|| =$$

A vector whose length is 1 is called a ______________________________.

- If we *divide* a nonzero vector $\mathbf{v}$ by its length − that is, multiply by _______________ − we obtain a unit vector $\mathbf{u}$.
- The process of creating $\mathbf{u}$ from $\mathbf{v}$ is called __________________________________ $\mathbf{v}$.
- We say that $\mathbf{u}$ is ____ as $\mathbf{v}$.

Example: Let $\mathbf{v} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$. Find a vector $\mathbf{u}$ in $\mathbb{R}^2$ of length 1 that is *in the same direction* as $\mathbf{v}$.

Distance in $\mathbb{R}^n$

Definition: For $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v**, denoted $\text{dist}(\mathbf{u}, \mathbf{v})$, is the length of the vector _______________. That is,

$$\text{dist}(\mathbf{u}, \mathbf{v}) =$$

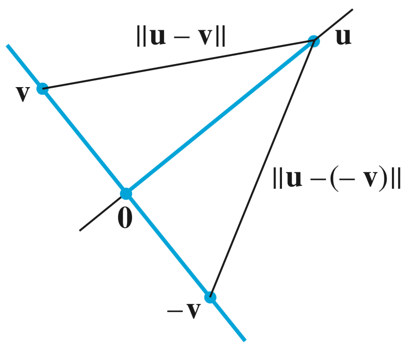
Remark: This is the usual formula for distance in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Example: Compute the distance between the vectors $\mathbf{u} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Orthogonal Vectors

Example: Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Find $[\text{dist}(\mathbf{u}, \mathbf{v})]^2$ and $[\text{dist}(\mathbf{u}, -\mathbf{v})]^2$.

Consider $\mathbb{R}^2$ or $\mathbb{R}^3$ and two lines through the origin determined by vectors $\mathbf{u}$ and $\mathbf{v}$ as shown in the figure below. Notice that these lines are perpendicular if and only if the distance from $\mathbf{u}$ to $\mathbf{v}$ is the same as the distance from $\mathbf{u}$ to $-\mathbf{v}$ (which is the same as requiring the squares of the distances to be the same).



Use your answers from the example above to determine when two lines are perpendicular to each other.

> **Definition:** Two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$ are **orthogonal** (to each other) if ______________.

> **Theorem 6.2 (The Pythagorean Theorem):** Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if
>
> $$||\mathbf{u} + \mathbf{v}||^2 =$$

Example: (Exercise 24.) Verify the *parallelogram law*, shown below, for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$.

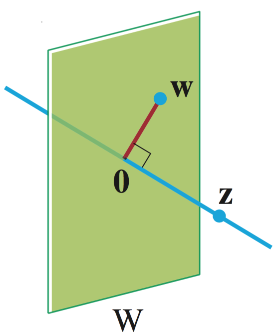
$$||\mathbf{u} + \mathbf{v}||^2 + ||\mathbf{u} - \mathbf{v}||^2 = 2||\mathbf{u}||^2 + 2||\mathbf{v}||^2$$

Orthogonal Complements

Definition: If a vector $\mathbf{z}$ is orthogonal to every vector in a subspace W of $\mathbb{R}^n$, then $\mathbf{z}$ is said to be ____.

The set of all vectors $\mathbf{z}$ that are orthogonal to W is called the ______________________ ______________________ of W, denoted by $W^{\perp}$.

Example 6. (pg. 336 in text) Let W be a plane through the origin in $\mathbb{R}^3$ and let L be the line through the origin and perpendicular to W. Suppose that $\mathbf{z}$ and $\mathbf{w}$ are nonzero, $\mathbf{z}$ is on L, and $\mathbf{w}$ is in W.



Notice that $\mathbf{z} \cdot \mathbf{w} =$ ______, so $\mathbf{z}$ and $\mathbf{w}$ are ________________ to each other. In fact, every vector on L is ________________ to every vector in W. That is,

$$L = \text{\_\_\_\_\_\_} \qquad \text{and} \qquad W = \text{\_\_\_\_\_\_}$$

Remark: Some important facts:

- A vector $\mathbf{x}$ is in $W^{\perp}$ if and only if $\mathbf{x}$ is orthogonal to ____________________________ in a spanning set for W.
- $W^{\perp}$ is a ______________________ of $\mathbb{R}^n$.

Theorem 6.3: Let A be an $m \times n$ matrix. Then

$$(\text{Row } A)^{\perp} = \qquad\qquad\qquad \text{and} \qquad\qquad (\text{Col } A)^{\perp} =$$

Proof: