Learning Objectives

- Determine if a matrix is diagonalizable
- Understand how to diagonalize a matrix

> **Definition:** A square matrix is ______________________ if A is similar to a diagonal matrix, that is, if ______________ for some ________________ matrix P and some ______________ matrix D.

When is A Diagonalizable?

The previous example had $A = PDP^{-1}$ with $A = \begin{bmatrix} 0 & 2 \\ -4 & 6 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, and $D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$. Compute

$$\begin{bmatrix} 0 & 2 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \qquad\qquad \begin{bmatrix} 0 & 2 \\ -4 & 6 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} =$$

What does this tell you about $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ with respect to A?

What can you conclude about how P and D are constructed?

Verify that another diagonalization of A is $A = P_1 D_1 P_1^{-1}$, where $P_1 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$.

> **Theorem 5.5 (The Diagonalization Theorem):**
>
> An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent ________________. In fact, $A = PDP^{-1}$, with D a diagonal matrix, if and only if the ______________ of P are n linearly independent ________________ of A. In this case, the ______________ entries of D are ______________ of A that correspond, respectively, to the eigenvectors in P.

Remark: A is diagonalizable if and only if A has enough eigenvectors to form a basis of ______. We call such a basis an **eigenvector basis** of ______.

Diagonalizing Matrices

Step 1. Find the eigenvalues of A.
Step 2. Find n linearly independent eigenvectors of A (if possible).
Step 3. Construct P from the vectors in Step 2.
Step 4. Construct D from the corresponding eigenvalues.

Theorem 5.6: An $n \times n$ matrix with ____ is diagonalizable.

Remark: The previous theorem provides a *sufficient* condition for a matrix to be diagonalizable. However, it is not *necessary* for an $n \times n$ matrix to have n distinct eigenvalues in order to be diagonalizable.

Matrices Whose Eigenvalues Are Not Distinct

Theorem 5.7: Let A be an $n \times n$ matrix whose distinct eigenvalues are $\lambda_1, \ldots, \lambda_p$.

(a) For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the multiplicity of the eigenvalue λ_k.

(b) The following statements are equivalent. That is, they are either all true or all false:

- (i) A is diagonalizable.
- (ii) The sum of the dimensions of the distinct eigenspaces of A equals n.
- (iii) For each $1 \leq k \leq p$, the dimension of the eigenspace for λ_k equals the multiplicity of λ_k.

(c) If A is diagonalizable and $\mathcal{B}_k$ is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\mathcal{B}_1, \ldots, \mathcal{B}_p$ forms an eigenvector basis for $\mathbb{R}^n$.