Learning Objectives

- Understand how to find the characteristic equation of a matrix
- Understand how to find the eigenvalues of a matrix

How Do We Find Eigenvalues?

Let A be an $n \times n$ matrix.

- λ is an eigenvalue of A if and only if the equation __________________ has a nontrivial solution.
- By the invertible matrix theorem, this happens if and only if the matrix $(A - \lambda I)$ is **not** ________________________.
- A matrix is **not** invertible if and only if the determinant of that matrix is equal to _______.
- Therefore, to find the eigenvalues of A, we must find all scalars λ such that

$$\det(A - \lambda I) = \_\_\_\_\_\_.$$

Example: Find the eigenvalues of $A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$.

Remark: If $\lambda = 0$ is an eigenvalue of A, then $\det(A - \lambda I) = 0$ simplifies to __________________ and we can conclude that A is **not** ________________________.

The Characteristic Equation

Definition: The equation ________________________ is called the **characteristic equation** of A. A scalar λ is an eigenvalue of a matrix A if and only if λ satisfies the characteristic equation of A.

If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a ____________________ of degree _______ called the **characteristic polynomial** of A.

The **multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic equation.

Example: The characteristic polynomial of a 6×6 matrix is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues of the matrix and their multiplicities.

Remarks:

- Because the characteristic equation for an $n \times n$ matrix involves an nth degree polynomial, the equation has exactly n roots, counting multiplicities, provided that complex roots are allowed.
- In practical work, eigenvalues of any matrix larger than a 2×2 should be found by a computer unless the matrix is triangular or has other special properties.

Similarity

Definition: Let A and B be $n \times n$ matrices.

A and B are **similar** if there is an ____________________ matrix P such that

______________ = _______ or, equivalently, _______ = ______________

Theorem 5.4: If $n \times n$ matrices A and B are similar, then they have the same characteristic polynomial, and hence the same __________________________ (with the same multiplicities).

Warnings:

- Matrices with the same eigenvalues do **not** have to be similar. For example, the matrices

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

  are **not** similar even though they have the same eigenvalues.

- Similarity is not the same as row equivalence. Row operations on a matrix usually change its eigenvalues.