Learning Objective

- Understand how to find the dimension of a finite-dimensional vector space
- Understand how to find the dimension of Nul A and Col A

The Dimension of a Vector Space

Theorem 4.9: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set in V containing more than ______ vectors must be ________________________________.

Theorem 4.10: If a vector space V has a basis of n vectors, then **every** basis of V must consist of **exactly** _______________________.

Definition: If V is a vector space spanned by a finite set, then V is said to be **finite-dimensional** in which case, the ____________________ of V, written as _________, is the number of vectors in ____________________________.

If V is not spanned by a finite set, then V is **infinite-dimensional**.

Example: The dimension of the zero vector space $\{\mathbf{0}\}$ is defined to be _________.

Example: What is the dimension of $\mathbb{R}^n$? Justify your answer. (*Hint:* What is the standard basis for $\mathbb{R}^n$?)

Example: What is the dimension of $\mathbb{P}_2$? Justify your answer. (*Hint:* What is the standard basis for $\mathbb{P}_n$?)

Subspaces of a Finite-Dimensional Space

The following theorem is a counterpart to the Spanning Set Theorem.

Theorem 4.11: Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to __________________________. In addition, H is finite-dimensional and

$$\dim H \quad \leq \quad \underline{\hspace{3cm}}$$

Theorem 4.12 (The Basis Theorem): Let V be a p-dimensional vector space, $p \geq 1$.

- Any ____ of exactly p elements in V is automatically a basis for V.
- Any set of exactly p elements that ________________ is automatically a basis for V.

The Dimensions of Nul A and Col A

Example: Suppose $A = \begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix}$. Find $\dim \text{Nul } A$ and $\dim \text{Col } A$.

You may use the fact that

$$\begin{bmatrix} 1 & 2 & -5 & 11 & -3 \\ 2 & 4 & -5 & 15 & 2 \\ 1 & 2 & 0 & 4 & 5 \\ 3 & 6 & -5 & 19 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 5 \\ 0 & 0 & 5 & -7 & 8 \\ 0 & 0 & 0 & 0 & -9 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\dim \text{Nul } A =$ _____________ and $\dim \text{Col } A =$ _____________

Summary:

- $\dim \text{Nul } A =$ ____
- $\dim \text{Col } A =$ ____